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# On the Relative K(2) of Non Commutative Local Rings.

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# ON THE RELATIVE $K_2$ OF NON-COMMUTATIVE LOCAL RINGS

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# ABSTRACT

This dissertation examines topics in Algebraic K-Theory, concerning the computation of absolute and relative Milnor groups, $K_2(R)$ and $K_2(R,I)$, for both commutative and non-commutative classes of rings, including the relative $K_2$ of non-commutative (not necessarily commutative) rings, and the absolute $K_2$ of commutative semilocal rings

Our main theorem is a natural extension of a result by Maazen and Stienstra [H Maazen and J Stienstra, A presentation of $K_2$ of split radical pairs, *J Pure Appl Algebra* **10**(1977), 271-294] which determines the relative $K_2$ of rings in a commutative setting We prove the non-commutative analog of this result for local rings

Other results proved in this dissertation include the redundancy of two relations given in Dennis and Stein's presentation for $K_2$ of a discrete valuation ring [R K Dennis and M R Stein, $K_2$ of discrete valuation rings, *Adv Math* **18**(1975), 182-238] and a proof that a normal form used by Kolster [M Kolster, $K_2$ of Non-Commutative Local Rings, *J Algebra* **95**(1985), 173-200] does not apply more generally to semilocal rings

## INTRODUCTION

The study of Algebraic K-Theory is closely related to the study of matrices over rings. As a whole, it is the study of the functors $K_0, K_1, K_2, \ldots$ from rings to groups and derives its name from the notation of these functors originally chosen by A. Grothendieck. The functor $K_0$ deals with the Grothendieck groups $K_0(R)$ consisting of isomorphism classes of finitely generated projective modules over a ring R. The functor $K_1$ deals with the Whitehead groups $K_1(R)$, that is, the factor group of the general linear group GL(R) by its elementary subgroup E(R), the subgroup generated by elementary matrices with entries in R.

The functor $K_2$ deals with the Milnor groups $K_2(R)$ that describe relations among the generators of the elementary group E(R). In fact, it reflects the relations of E(R) which arise due to the choice of R - not one of the "Steinberg relations" which hold in E(R) for any R. More explicitly, we define $K_2(R)$ by the exact sequence

$$1 \longrightarrow K_2(R) \longrightarrow St(R) \longrightarrow E(R) \longrightarrow 1,$$

where the Steinberg group, St(R), is an abstract group with generators similar to the generators of E(R), but freer in the sense that its only relations are the three classes of Steinberg relations which hold in E(R) for all R. Thus, intuitively, we may describe $K_2(R)$ as the set of all nontrivial relations in E(R).

The functors $K_3$, $K_4$, ... are highly motivated by topological concerns. In this dissertation, we concentrate on $K_2$, and are interested in the other K-groups as they relate to $K_2$ through a Mayer-Vietoris type long exact sequence involving the relative K-groups. Chapters I and II provide background information about absolute and relative $K_2$, and about the Mayer-Vietoris long exact sequence.

Chapter III presents some known results concerning the calculation of $K_2$ for some specific classes of rings. Following Matsumoto's presentation for $K_2$ of a field using Steinberg symbols [HM], there have been further attempts to generalize this result to more general classes of rings. Some of the important contributions include the Dennis-Stein presentation of $K_2$ for a discrete valuation ring [DS2], and Rehmann's result for skew fields [UR], which gives $K_2$ up to a group extension, as do all current computations of $K_2$ for non-commutative rings. Kolster [MK2] demonstrates some very general results for $K_2$ in the non-commutative case, but at the cost of using (generalized Dennis-Stein) symbols of greater "length". In the case of non-commutative local rings, however, he is able to sharpen the result to standard Dennis-Stein symbols [MK3].

In Chapter IV, following a suggestion by Kolster in [MK2], we show that two of the longer relations in the Dennis-Stein presentation of a discrete valuation ring are redundant.

Given Kolster's determination of $K_2$ of a non-commutative local ring, one may naturally wonder whether the normal form for elements of $K_2(R)$ can be applied successfully to the semilocal case. The answer is negative, as we show in Chapter V by counter-example.

The main result of this dissertation is an extension of a theorem by Maazen and Stienstra that computes the relative $K_2$ of a commutative ring. Maazen and Stienstra [MS] have determined $K_2$ of split radical pairs, i.e. $K_2$ of a commutative ring, R, relative to a radical ideal I, such that the canonical projection, $R \longrightarrow R/I$, splits. The presentation given is

$$K_2(R,I) \cong D(R,I),$$

where $D(R,I)$ is the abelian group generated by Dennis-Stein symbols $\langle a,b \rangle$, with $(a,b) \in R\times I \cup I\times R$, and subject to a set of three standard relations.

Our main theorem is the analogous non-commutative result for local rings, and is proved in Chapter VI. Specifically, let R be a local ring, not necessarily commutative, and I be a split proper ideal (i.e. $R \longrightarrow R/I$ splits).

Then the sequence

$$1 \longrightarrow K_2(R,I) \longrightarrow D^{+}(R,I) \longrightarrow [R^{*},1+I] \longrightarrow 1$$

is exact, where we define $D^{+}(R,I)$ to be the (usually non-abelian) group generated by Dennis-Stein symbols $\langle a,b\rangle$ with $R\times I \cup I\times R$, and subject to a set of 18 relations. These relations reduce in the commutative case to the three relations used by Maazen-Stienstra and a fourth implying that $D^{+}(R,I)$ is abelian, thus yielding the Maazen-Stienstra result as a special case. It should be noted that a theorem by Keune [FK2] proves that Maazen-Stienstra's theorem holds also in the non-split case. A similar extension of our result to the non-split case for the relative $K_2$ of non-commutative local rings is expected to hold.

# CHAPTER I: Steinberg Groups and $K_2$

The (unstable) Steinberg group St(n,R) is a group modeled after the group generated by elementary matrices over a ring, R, so we begin by defining the general linear group, GL(n,R), and the elementary group, E(n,R)

**Note**: By ring, we shall always mean an associative ring with 1.

**Definition 1.1**: Let R be a ring. Then **GL(n,R)** is the multiplicative group of $n \times n$ invertible matrices with entries in R

**Definition 1.2**: **E(n,R)** is the subgroup of GL(n,R) generated by the matrices $e_{ij}(a)$, $i \neq j$, where

$$e_{ij}(a) = (b_{rs}), \quad b_{rs} = \begin{cases} 1, & \text{if } r=s \\ a, & \text{if } r=i \text{ and } s=j \\ 0, & \text{otherwise} \end{cases}$$

Two important classes of matrices in the elementary group are

$$W_{ij}(u) = e_{ij}(u)e_{ji}(-u^{-1})e_{ij}(u), \quad u \in R^*$$

and
$$H_{ij}(u) = W_{ij}(u)W_{ij}(-1), \quad u \in R^*.$$

The matrix $H_{ij}(u)$ is a diagonal matrix with 1's on the diagonal except at locations ii and jj which are $u$ and $u^{-1}$ respectively.

By matrix multiplication, we may easily check that the following relations are satisfied in E(n,R) for $n \geq 3$

(E1) $e_{ij}(a)e_{ij}(b) = e_{ij}(a+b)$

$$(E2)\quad [e_{ij}(a),e_{kl}(b)] = \begin{cases} 1, & \text{if } j\neq k \text{ and } i\neq l \\ e_{il}(ab), & \text{if } j=k \text{ and } i\neq l \end{cases}$$

In the case n=2, E2 becomes a trivial consequence of E1, so for n=2, we choose to consider the two relations

$$(E1)\quad e_{12}(a)e_{12}(b) = e_{12}(a+b)$$
$$(E2')\quad W_{12}(u)e_{21}(a)W_{12}(u)^{-1} = e_{12}(-uau)$$

These relations hold for any ring, R, and will be the basis for our definition of St(n,R)

**Definition 1.3**: The (unstable) **Steinberg group** St(n,R), for $n\geq 2$, is defined to be the group with generators $x_{ij}(a)$, $a\in R$, $i,j=1,\ldots,n$, $i\neq j$, and relations

$$(S1)\quad x_{ij}(a)x_{ij}(b) = x_{ij}(a+b)$$
$$(S2)\quad [x_{ij}(a),x_{kl}(b)] = \begin{cases} 1, & \text{if } j\neq k \text{ and } i\neq l \\ x_{il}(ab), & \text{if } j=k \text{ and } i\neq l \end{cases}$$

In the case n=2, we replace relation S2 with S2' and the defining relations are

$$(S1)\quad x_{12}(a)x_{12}(b) = x_{12}(a+b)$$
$$(S2')\quad w_{12}(u)x_{21}(a)w_{12}(u)^{-1} = x_{12}(-uau),$$

where $w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u)$.

While the case n=2 may appear disjoint, it is seen to be natural by noting that S1 and S2 imply S2' when $n\geq 3$.

Clearly, we have a homomorphism defined on generators by

$$\Phi:\ St(n,R) \longrightarrow E(n,R)$$

$$x_{ij}(a) \longmapsto e_{ij}(a)$$

This homomorphism, $\phi$, is called the (unstable) **Steinberg map**. Although there is one map for each $n \geq 2$, we refer to each of them as "the" Steinberg map.

Just as $x_{ij}(a)$ in the Steinberg group is an obvious analog to the elementary matrix $e_{ij}(a)$, the matrix $w_{ij}(u)$ is the Steinberg group analog to its image $W_{ij}(u)$ in the elementary group. We similarly define the analog of the diagonal matrix $H_{ij}(u)$ to be

$$h_{ij}(u) = w_{ij}(u)w_{ij}(-1), \quad u \in R^*$$

The Steinberg map, of course, maps $h(u)$ to $H(u)$.

We are now in a position to define the (unstable) $K_2$ of a ring.
**Definition 1.4**: Let $R$ be a ring and $n \geq 2$. Then $\mathbf{K_2(n,R)}$ is defined to be the kernel of the Steinberg map, i.e. it is defined by the exact sequence:

$$1 \longrightarrow K_2(n,R) \longrightarrow St(n,R) \longrightarrow E(n,R) \longrightarrow 1$$

To each of the unstable groups $GL(n,R)$, $E(n,R)$, $St(n,R)$, and $K_2(n,R)$, there corresponds a stable group denoted $GL(R)$, $E(R)$, $St(R)$, and $K_2(R)$ respectively, which are defined as direct limits.

**Definition 1.5**: Let $I$ be a set. A **partial order** on $I$ is a relation, $\leq$, satisfying the following conditions:

(1) For all $i,j,k \in I$, we have $i \leq i$
(2) If $i \leq j$ and $j \leq k$ then $i \leq k$
(3) If $i \leq j$ and $j \leq i$ then $i = j$.

We say that $I$ is **directed** if given $i,j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$

Let $\{A_i\}$ be a family of objects in a category $\mathcal{C}$, and indexed by a directed set, $I$. For each pair $i, j \in I$ such that $i \le j$, assume there is a morphism

$$f_j^i : A_i \longrightarrow A_j$$

such that whenever $i \le j \le k$, we have

$$f_k^j f_j^i = f_k^i \text{ and } f_i^i = \mathrm{id}.$$

Then a **direct limit** for the family $\{f_j^i\}$ is a universal object consisting of a pair $(A, (f^i))$, where $A \in \mathrm{Ob}(\mathcal{C})$ and $(f^i)$ is a family of morphisms

$$f^i : A_i \longrightarrow A, \qquad i \in I$$

such that the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & A_i \arrow[r, "f_j^i"] \arrow[dr, "f^i"'] & A_j \arrow[r] \arrow[d, "f^j"] & \cdots \\ & & A & \end{tikzcd}$$

We define $GL(R)$ to be the direct limit of the $GL(n,R)$, and $E(R), St(R), K_2(R)$ to be the direct limits of $E(n,R)$, $St(n,R)$, $K_2(n,R)$ respectively. $GL(R)$ may be viewed as the union of the $GL(n,R)$ where $GL(n,R)$ is included into $GL(n+1,R)$ by

$$f_{n+1}^n : A \longmapsto \begin{pmatrix} A & \\ & 1 \end{pmatrix}.$$

For $E(R)$, $St(R)$, and $K_2(R)$, we have canonical maps

$$E(n,R) \longrightarrow E(n+1,R)$$
$$St(n,R) \longrightarrow St(n+1,R)$$

$$K_2(n,R) \longrightarrow K_2(n+1,R)$$

Furthermore, we still have a Steinberg map

$$\begin{aligned} \phi: \quad St(R) &\longrightarrow E(R) \\ x_{ij}(a) &\longmapsto e_{ij}(a) \end{aligned}$$

and $K_2(R)$ is the kernel of the Steinberg map.

We also note that for a ring, R, $K_2(R)$ may be characterized as the center of St(R), and St(R) may be characterized as the universal central extension of E(R).

In some cases, it may be useful to consider modifications of the standard Steinberg groups we have just defined. For example, stability theorems of Kolster [MK1] show that when R is semilocal (or any ring which satisfies Bass' stable range condition $SR_2$), $K_2(R)$ is isomorphic to the groups $K_2(n,R)$ for $n \geq 3$, and (for n=2) isomorphic to the modified group $K_2'(2,R)$ which is defined as follows.

**Definition 1.6: $K_2'(2,R)$** is defined by the exact sequence

$$1 \longrightarrow K_2'(2,R) \longrightarrow St'(2,R) \longrightarrow E(2,R) \longrightarrow 1.$$

**Definition 1.7: St'(2,R)** is the quotient group St(2,R)/W(2,R), where W(2,R) is the normal subgroup of St(2,R) generated by the elements:

(a) $tx_1(a)t^{-1}x_1(-ua)$ and $tx_2(a)t^{-1}x_2(-au^{-1})$,
where $t \in S(1,R)$, (defined below)
and t maps to the matrix $\begin{pmatrix} u & \\ & 1 \end{pmatrix}$

(b) $x^y x_y^{-1}$, where $(x^y, x_y)$ is a y-pair with $x^y, x_y \in S(1,R)$.

**Definition 1.8** **S(1,R)** is defined to be the inverse image of $E(2,R) \cap GL(1,R)$ in $St(2,R)$, under the Steinberg map.

**Definition 1.9**: A **y-pair** in $St(2,R)$ consists of two elements

$$x^y = \rho\Pi(x_1(a_i y)x_2(b_i))$$
$$x_y = \rho\Pi(x_1(a_i)x_2(y b_i))$$

with $\rho \in S(1,R)$

We have used the indices 1 and 2 for 12 and 21 respectively.
We finish this section by listing some useful identities which hold in $St'(2,R)$

**Theorem 1.10**. The following identities hold in $St'(2,R)$

(1) $w_2(u) = w_1(-u^{-1})$, $u \in R^*$,
(2) $h_2(u) = h(u)^{-1}$, $u \in R^*$,
(3) $x_2(a)x_2(b) = x_2(a+b)$, $a, b \in R$,
(4) $w(u)x_2(a)w(u)^{-1} = x_1(-uau)$, $a \in R$, $u \in R^*$,
(5) $w(u)w(v)w(u)^{-1} = w(uv^{-1}u)$, $u, v \in R^*$,
(6) $h(u)w(v)h(u)^{-1} = w(uvu)$, $u, v \in R^*$,
(7) $\langle a,b\rangle_2 = \langle a,b\rangle h(\theta^{-1})$, $a, b \in R$, $1+ab \in R^*$, $\theta=(1+ab)(1+ba)^{-1}$,

**Proof**: See [MK2]. We have modified the statements slightly by using the first 2 relations to write each identity in terms of $w(u) = w_1(u)$ instead of both $w_1(u)$ and $w_2(u)$, and $h(u) = h_1(u)$ instead of both $h_1(u)$ and $h_2(u)$. Otherwise, the proofs are as in [MK2].

$\square$

# CHAPTER II: The Mayer-Vietoris Long Exact Sequence

The Mayer-Vietoris long exact sequence of Algebraic K-Theory is an exact sequence which is not only obviously similar in form to the Mayer-Vietoris sequence of Algebraic Topology, but also shares its fundamental importance. In fact, the Mayer-Vietoris sequence motivates the choice of definition for the relative K-functors. One early definition of relative $K_2$ by Milnor was shown by Swan [SW] to be less desirable by proving the long exact Mayer-Vietoris sequence involving Milnor's relative $K_2$ was impossible no matter what definition for the relative $K_3$ functor is chosen.

In this paper we use a definition of relative $K_2$ which was first introduced by Stein [ST1]. This "correct" relative Steinberg group, $St(R,I)$, is defined to be the 0th relative derived functor of $St(R)$ relative to the congruence relation in a ring $R$ corresponding to an ideal, $I$. (See [FK1] and [FK2]) Keune, in [FK2], proves the equivalence of the characterization which we will take as our definition.

**Definition 2.1**: Let $R$ be a ring with two-sided ideal, $I$. Then the **fibered product**, $R(I)_1$, is defined to be

$$R(I)_1 = \{(x_0, x_1) \in R\times R \text{ such that } \pi(x_0) = \pi(x_1)\}$$

where $\pi: R \longrightarrow R/I$ is the canonical projection.

Now consider the Steinberg group, $St(R(I)_1)$. In particular, we are interested in the elements $x_{ij}(r,r)$ and the elements $y_{ij}(a)$ which we define as

$$y_{ij}(a) = x_{ij}(0,a)$$

Keune first gives a presentation of the kernel of the map induced by $p_1$, deletion of the first component,

$$\mathrm{Ker}(\mathrm{St}(R(I)_1) \longrightarrow \mathrm{St}(R))$$

as a St(R)-group as the following:

Generators are symbols $y_{ij}(a)$, where $a \in I$, $i \neq j$
Defining relations are

| | | |
|---|---|---|
| (1) | $y_{ij}(a)y_{ij}(b) = y_{ij}(a+b)$ | |
| (2) | $[y_{ij}(a), y_{kl}(b)] = 1$ | if $i \neq l$ and $j \neq k$ |
| (3) | $[y_{ij}(a), y_{jk}(b)] = y_{ik}(ab)$ | if $i \neq k$ |
| (4) | $x_{ij}(r)y_{ij}(a) = y_{ij}(a)$ | |
| (5) | $x_{ij}(r)y_{kl}(a) = y_{kl}(a)$ | if $i \neq l$ and $j \neq k$ |
| (6) | $x_{ij}(r)y_{jk}(a) = y_{ik}(ra)y_{jk}(a)$ | if $i \neq k$ |
| (7) | $x_{ij}(r)y_{ki}(a) = y_{kj}(-ar)y_{ki}(a)$ | if $k \neq j$ |

where a generator $x_{ij}(r)$ of St(R) operates on $\mathrm{Ker}(p_1)$ by conjugation with $x_{ij}(r,r)$ in $\mathrm{St}(R(I)_1)$.

**Definition 2.2**: **St(R,I)** is (as an St(R) group) defined by generators $y_{ij}(a)$, where $a \in I$, $i \neq j$, and relations (1)-(7) above, together with the relation

$$x_{12}(b)y_{21}(a) = y_{12}(b)y_{21}(a)y_{12}(-b).$$

Under this presentation, we may define the Steinberg map on generators in the natural manner as

$$\begin{aligned} \Phi:\ \mathrm{St}(R,I) &\longrightarrow E(R,I) \\ y_{ij}(a) &\longrightarrow e_{ij}(a). \end{aligned}$$

**Definition 2.3**: **$K_2$(R,I)** is the kernel of the Steinberg map, i.e. is defined by the exact sequence

$$1 \longrightarrow K_2(R,I) \longrightarrow St(R,I) \longrightarrow E(R,I) \longrightarrow 1$$

This definition along with proper definitions for the other K-groups (see [SG2]) gives rise to the exactness of the Meyer-Vietoris long exact sequence:

$$\ldots \longrightarrow K_3(R) \longrightarrow K_3(R/I) \longrightarrow K_2(R,I) \longrightarrow K_2(R) \longrightarrow K_2(R/I)$$
$$\longrightarrow K_1(R,I) \longrightarrow K_1(R) \longrightarrow K_1(R/I) \longrightarrow K_0(R,I) \longrightarrow \ldots$$

An important situation to be considered in the relative case is when the canonical projection $R \longrightarrow R/I$ splits.

**Definition 2.4**. Let $R$ be a ring with a two-sided ideal, $I$. Then the homomorphism $\pi: R \longrightarrow R/I$ is said to **split** if it admits a section, i.e. if there exists an (injective) group homomorphism $s$: $R/I \to R$ such that $\pi s = id_{R/I}$, the identity on $R/I$.

Clearly, if $R \longrightarrow R/I$ splits, then as a group, we have $R \cong R/I \oplus I$, and so every element $r \in R$ may be written uniquely as $r = a+i$, with $a \in s(R/I)$, $i \in I$. We often identify $s(R/I)$ and $R/I$ so that we may refer to $R/I$ as a subgroup, i.e. $R/I \subset R$ as a subgroup of $R$.

In the case when $R \longrightarrow R/I$ splits (compare [FK2]), it follows as a consequence of the Mayer Vietoris sequence that

$$K_2(R,I) = \ker[K_2(R) \longrightarrow K_2(R/I)].$$

This is a very natural result. In fact, in [SY], Sylvester chooses this for his definition of relative $K_2$ and shows how it may be used in the calculation of $K_2$. Many of the results for relative $K_2$ are proved in the split case, and some (including Maazen-Steinstra's presentation for $K_2$ of split radical pairs[MS] and Swan's excision [SW]) may be easily extended to the non-split case as shown by Keune [FK2].

# CHAPTER III: Presentations of $K_2$

In this chapter, we introduce some previous results which lead naturally to the new theorems and observations given in this dissertation. Stated very broadly, we are interested in the determination of the absolute and relative $K_2$ of certain classes of rings. In the case of commutative rings, R, we look for a presentation of the absolute K-group, $K_2(R)$, or the relative K-group, $K_2(R,I)$, relative to an ideal, I, in terms of generators and relations. In the non-commutative case, we want a presentation for a group $D(R)$ or $D^+(R,I)$ in the absolute and relative cases respectively, such that the appropriate sequence is exact

$$1 \longrightarrow K_2(R) \longrightarrow D(R) \longrightarrow [R^*,R^*] \longrightarrow 1$$

or as we generalize to the relative situation in Chapter VI,

$$1 \longrightarrow K_2(R,I) \longrightarrow D^+(R,I) \longrightarrow [R^*,1+I] \longrightarrow 1$$

where R is local.

A fundamental result, whose generalization has led to most of the results we will consider, is Matsumoto's presentation for $K_2(F)$ of a field, F, in terms of Steinberg symbols, i.e. with Steinberg symbols as generators of $K_2(F)$.

**Definition 3.1**: Given a ring, R, the **Steinberg symbol** $\{u,v\}_{ij}$, $u,v \in R^*$ is defined to be the element

$$\{u,v\}_{ij} = h_{ij}(uv)h_{ij}(u)^{-1}h_{ij}(v)^{-1}$$

in the Steinberg group, $St(R)$.

If the ring, R, is commutative, then we may omit the indices, because the symbol is independent of the indices i and j [JM] When we are considering the unstable or the modified Steinberg groups (usually modified in the sense of adding another relation or set of relations as in $St'(R)$ - see Definition 1 7) we still have Steinberg symbols defined in the same manner

In the unstable, non-commutative case with n=2, we may write Steinberg symbols in St(2,R), or its modified versions, as $\{u,v\}_1$ and $\{u,v\}_2$ for $\{u,v\}_{12}$ and $\{u,v\}_{21}$ respectively

**Theorem 3.2** (Matsumoto,1969) If R is a field, then $K_2(R)$ has the following presentation as an abelian group

| | | |
|---|---|---|
| Generators | $\{u,v\}$ | $(u,v \in R^*)$ |
| Relations | $\{u_1u_2,v\} = \{u_1,v\}\{u_2,v\}$ | |
| | $\{u,v_1v_2\} = \{u,v_1\}\{u,v_2\}$ | |
| | $\{u,1-u\} = 1,\ u \neq 1$ | |

Note that we have not distinguished notationally the Steinberg symbols existing in the Steinberg group from the abstract symbols which are generators in the presentation In fact, the theorem should be interpreted as an isomorphism between $K_2(F)$, which is the subgroup generated by Steinberg symbols in the Steinberg group, and the group defined by the abstract symbols $\{u,v\}$, $u,v \in F$, as generators and subject to the three relations This distinction between the two sets of symbols should by clear from context, so we write them identically

We also point out that Matsumoto's Theorem may be restated in terms of a tensor product over $\mathbb{Z}$

**Theorem** 3 2 (Restated) If R is a field, then

$$K_2(R) \cong (R^* \otimes R^*) \,/\, \langle\, u \otimes (1-u) \mid u \in R^*,\ u \neq 1 \rangle$$

The following corollary of Matsumoto's Theorem is a simple consequence of elementary properties of finite fields, and the fact that Steinberg symbols generate $K_2(F)$ of a field and are subject to the relations of Matsumoto's Theorem

**Corollary 3.3** Let F be a finite field Then $K_2(R)$ is trivial
**Proof** This is an easy exercise, see [SY]

$\square$

Dennis and Stein have generalized Matsumoto's theorem in the commutative case to discrete valuation rings with the following [DS2]

**Theorem 3.4** (Dennis-Stein, 1975) Let R be a commutative discrete valuation ring with maximal ideal P, then $K_2(R)$ is isomorphic to $S(R)$, the abelian group generated by the Steinberg symbols $\{u,v\}$, $u,v \in R^*$, subject to the relations

(S1) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$

(S2) $\{u,v\} = \{v,u\}^{-1}$

(S3) $\{u,-u\} = 1$

(S4) $\{u, 1-u\} = 1$, if $1-u \in R^*$

(S5) $\{v, 1-pqv\} = \{-(1-qv)(1-p)^{-1}, (1-pqv)(1-p)^{-1}\}$
$\qquad\qquad\{-(1-pv)(1-q)^{-1}, (1-pqv)(1-q)^{-1}\}$

(S6) $\{-(1-qr)(1-p)^{-1}, (1-pqr)(1-p)^{-1}\}$
$\qquad\{-(1-pr)(1-q)^{-1}, (1-pqr)(1-q)^{-1}\}$
$\qquad\{-(1-pq)(1-r)^{-1}, (1-pqr)(1-r)^{-1}\} = 1$

(S7) $\{u_1, 1+qu_1\}\{u_2(1-qu_1)^{-1}, [1+q(u_1+u_2)](1+qu_1)^{-1}\}$
$\qquad= \{v_1, 1+qv_1\}\{v_2(1-qv_1)^{-1}, [1+q(v_1+v_2)](1+qv_1)^{-1}\}$

for all $p,q,r \in P$, $u,v,u_1,u_2,v_1,v_2 \in R^*$ such that $u_1+u_2 = v_1+v_2 \in P$

It can be shown (see Chapter IV for a proof) that relations S5 and S6 are direct consequences of relations S1-S4 and S7

In the case of the principal ideal domains $\mathbf{Z}/m\mathbf{Z}$, Sylvester [SY] shows that $K_2(\mathbf{Z}/m\mathbf{Z})$ is isomorphic to $\mathbf{Z}/2\mathbf{Z}$ if $m \equiv 0 \pmod 4$, and trivial otherwise

So far, these results have been generalizations of Matsumoto's theorem in the commutative case On the other hand, Rehmann [UR] generalizes Matsumoto's Theorem to the case of a skew field But before stating this theorem, we first define Dennis-Stein symbols

**Definition 3.5** Let R be a ring, and a,b be elements of R such that $\epsilon = 1+ab \in R^*$, and $\bar{\epsilon} = 1+ba$ Then we define the **Dennis-Stein symbol**,

$$\langle a,b\rangle_{ij} = x_{ij}(a)x_{ji}(b)x_{ij}(-\epsilon^{-1}a)x_{ji}(-\bar{\epsilon}b)h_{ij}(\bar{\epsilon})^{-1},$$

where a,b are such that $\epsilon = 1+ab \in R^*$, and $\bar{\epsilon} = 1+ba$

These Dennis-Stein symbols generalize Steinberg symbols in the sense that in the Steinberg group we have the equality [SY]

$$\{u,v\}_{ij} = \langle u(v-1),u^{-1}\rangle_{ij}$$

Now we state Rehmann's determination of $K_2(R)$ for skew fields in terms of an exact sequence

**Theorem 3.6**(Rehmann, 1978) Given a skew field, R, there is a short exact sequence

$$1 \longrightarrow K_2(R) \longrightarrow D_0(R) \longrightarrow [R^*,R^*] \longrightarrow 1$$

where $D_0(R)$ is the group generated by $\{u,v\}$, $u,v \in R^*$, subject to the relations

(V1) $\{u, 1-u\} = 1$

(V2) $\{uv, w\} = {}^{u}\{v, w\}\{u, w\}$

(V3) $\{u, v\}\{v, u\} = 1$

and where we define ${}^{u}\{v, w\} = \{uvu^{-1}, uwu^{-1}\}$.

Furthermore, $D_0(R)$ is (for any local ring with residue class field, $R/\mathrm{rad}\, R$, not isomorphic to $\mathbb{F}_2$ - see [MK3]) isomorphic to $D_1(R)$, defined to be the group generated by $\langle a,b\rangle$, where $1+ab \in R^*$, subject to the relations

(R1) $\langle a,b\rangle\langle -b,-a\rangle = 1$

(R2) $\langle ay,b\rangle\langle ba,y\rangle = \langle a,yb\rangle$

(R3) $\langle a,b\rangle\langle c,d\rangle\langle a,b\rangle^{-1} = \langle \theta c\theta^{-1}, \theta d\theta^{-1}\rangle$, $\theta = (1+ab)(1+ba)^{-1}$

(R4) $\langle a,b\rangle\langle \varepsilon a\hat{\varepsilon}^{-1}, (c-b)\hat{\varepsilon}^{-1}\rangle = \langle a,c\rangle$, $\varepsilon = 1+ba$

Rehmann's result establishes an approach to determining $K_2(R)$ in the non-commutative case by finding $K_2(R)$ up to a group extension as a subgroup of the group $D_0(R)$ or $D_1(R)$ generated by Steinberg or (standard) Dennis-Stein symbols. In fact, the map

$$K_2(R) \longrightarrow D_0(R)$$

is defined on generators by

$$\{u,v\}_{ij} \longrightarrow \{u,v\}_{ij}$$

and similarly, we have

$$\begin{aligned} K_2(R) &\longrightarrow D_1(R) \\ \langle a,b\rangle_{ij} &\longrightarrow \langle a,b\rangle_{ij}. \end{aligned}$$

By exactness, we see that the elements of $K_2(R)$ are products of symbols $\prod\{u_i,v_i\}$ with $\prod[u_i,v_i] = 1$ or equivalently, $\prod\langle a_i,b_i\rangle$ with $\prod\theta_i = 1$

Kolster [MK2] generalizes the result by Rehmann to the case of arbitrary rings, but at the cost of using generalized symbols in place of the standard Dennis-Stein symbols. An invariant related to a measure of stability, the word-length $l(R)$, is defined. For example, a ring that has stable rank 1, i.e. satisfies Bass' stable range condition $SR_2$, (e.g. a semilocal ring) would have word-length 2. $K_2'(2,R)$, which behaves well under stabilization with respect to the linear $K_2$-groups, is then determined up to a group extension in terms of generalized Dennis-Stein symbols. (This reduces to a presentation of $K_2'(2,R)$ in the commutative case.)

**Theorem 3.7**(Kolster, 1984) Let $R$ be an arbitrary ring. For all $m \geq l(R)$ the groups $D_m$, $D_{m+1}$, $D_{m+2}$, ..., $D_\infty$ are isomorphic and the following sequence is exact

$$1 \longrightarrow K_2'(2,R) \longrightarrow D_m \longrightarrow E(2,R) \cap GL(1,R) \longrightarrow 1,$$

where the groups $D_n$ are defined by generators $\langle a_1, \ldots, a_{2n} \rangle_n$, Dennis-Stein symbols of length $2n$, and relations

(M1) $\langle a_1, \ldots, a_{2n} \rangle_n \langle -a_{2n}, \ldots, -a_1 \rangle_n = 1$

(M2) $\langle a_1 y, a_2, \ldots, a_{2n} \rangle_n = \langle a_1, ya_2, \ldots, ya_{2n} \rangle_n \langle p, y \rangle_n^{-1}$,
where $1 + py$ is the $\varepsilon$ associated to $\langle a_1 y, a_2, \ldots, a_{2n} \rangle_n$

(M3) $s(\langle a_1, \ldots, a_{2n} \rangle_n)s^{-1} = \langle \theta a_1 \theta^{-1}, \ldots, \theta a_{2n} \theta^{-1} \rangle_n$
where $\theta$ corresponds to $\langle a_1, \ldots, a_{2n} \rangle_n$,
and $s$ is a general symbol of length $2m$

(M4) $s(\varepsilon t \hat{\varepsilon}^{-1}) = z$,
where $s = \langle a_1, \ldots, a_{2n} \rangle_n$,
$z = \langle a_1, \ldots, a_n, b_{n+1}, \ldots, b_{2n} \rangle_n$,
if $n \geq 2$, $t = \langle -a_{2n+1}, -a_{2n}, \ldots, -a_{n+2}, -a_{n+1}+b_{n+1}, b_{n+2}, \ldots, b_{2n} \rangle_n$,
if $n = 1$, $t = \langle -a_3, b_2 - a_2 \rangle_1$

If $n \geq 2$, then we also need the fifth relation
(M5) $\langle 1,-1,q,1\rangle_n = 1$, for all $q \in R$

(We refer the reader to [MK2] for a precise definition of the terms $\theta, \varepsilon, a_{2n+1}$, in terms of the general symbols, but they are a straightforward generalization of the case of standard Dennis-Stein symbols )

The standard Dennis-Stein symbols are generalized symbols of length 2, and the group $D_1(R)$ is the same group $D_1(R)$ defined above in terms of standard Dennis-Stein symbols Of course, $D_0$ does not continue the pattern as a group of generalized symbols of length zero, but is an group similar to $D_1(R)$, generated by Steinberg symbols As we noted earlier, in the case of local rings with large enough residue class fields, $D_0(R)$ and $D_1(R)$ are isomorphic

**Corollary 3.8**(Kolster,1984) Let R be a ring with stable rank 1 Then the groups $D_2, \quad , D_\infty$ are isomorphic and the following sequence is exact

$$1 \longrightarrow K_2'(2,R) \longrightarrow D_2 \longrightarrow E(2,R)\cap GL(1,R) \longrightarrow 1$$

This corollary is therefore, in particular, a determination of $K_2(R)$ for local rings in terms of symbols of length 4, because in this case[MK1],

$$K_2'(2,R) \cong K_2(3,R) \cong \ldots \cong K_2(R)$$

This result is sharpened to standard (length 2) symbols as follows

**Theorem 3.9**(Kolster, 1985) If R is local, there is a short exact sequence

$$1 \longrightarrow K_2(R) \longrightarrow D_1(R) \longrightarrow [R^*,R^*] \longrightarrow 1,$$

where $D_1(P)$ is as defined above

The statement of the theorem also makes use of the equality

$$E(2,A) \cap GL(1,A) = [R^*,R^*]$$

to write the exact sequence in terms of a commutator subgroup In the commutative case, this theorem reduces to an isomorphism,

$$K_2(R) \cong D_1(R)$$

Taking a different approach, instead of determining $K_2(R)$ for wider classes of rings, Maazen and Stienstra[MS] generalize to the relative case, and give a presentation in the commutative case for $K_2$ of split radical pairs

**Definition 3.10** Let R be a ring and I be a two-sided ideal in R Then (R,I) is a **split radical pair** if I is a radical ideal, and the canonical projection, $R \longrightarrow R/I$, splits

**Theorem 3.11**(H Maazen,J Stienstra) If (R,I) is a split radical pair, then, for any integer $n \geq 3$, the homomorphism

$$\delta \; D(R,I) \longrightarrow K_2(n,R,I)$$

is an isomorphism Hence,

$$\delta \; D(R,I) \longrightarrow K_2(R,I)$$

is also an isomorphism,

where D(R,I) is the abelian group defined by generators $\langle a,b \rangle$, one for each couple $(a,b) \in R \times I \cup I \times R$ such that $1+ab \in R^*$

and relations
(D1) $\langle a,b \rangle \langle -b,-a \rangle = 1$
(D2) $\langle a,b \rangle \langle a,c \rangle = \langle a, b+c+abc \rangle$
(D3) $\langle a,bc \rangle = \langle ab,c \rangle \langle ac,b \rangle$.

Theorem 3.11 gives a presentation of $K_2(R,I)$ for commutative rings only. In this dissertation, we extend this result to some non-commutative local rings (see Chapter VI, Theorem 6.1). We also note here that the result of Theorem 3.11 does not depend upon whether or not the pair $(R,I)$ is a split pair. The proof that the split case implies the general case is given by Keune [FK2].

## CHAPTER IV: Dennis-Stein Presentation - Redundant Relations

As discussed in the previous chapter, one presentation for $K_2(R)$ in the commutative case is the Dennis-Stein presentation for $K_2$ of a discrete valuation ring. A list of seven relations S1 through S7 were given. Following a suggestion in [MK3], we show not only that S5 and S6 are redundant in the presence of the other relations S1-S4,S7, but also the direct manner in which S5 and S6 are consequences of the other relations.

Once again we state the Dennis-Stein presentation.

**(Original) Theorem 4.1** If R is a commutative discrete valuation ring with maximal ideal $P = \text{rad } R$, then $K_2(R) \cong S(R)$, where $S(R)$ is the abelian group generated by the Steinberg symbols $\{u,v\}$, $u,v \in R^*$ subject to the seven defining relations

$$\begin{array}{ll}
(S1) & \{u_1u_2,v\} = \{u_1,v\}\{u_2,v\} \\
(S2) & \{u,v\} = \{v,u\}^{-1} \\
(S3) & \{u,-u\} = 1 \\
(S4) & \{u,1-u\} = 1, \text{ if } 1-u \in R^* \\
(S5) & \{v,1-pqv\} = \{-(1-qv)(1-p)^{-1},(1-pqv)(1-p)^{-1}\} \\
& \qquad\qquad\qquad \{-(1-pv)(1-q)^{-1},(1-pqv)(1-q)^{-1}\} \\
(S6) & \{-(1-qr)(1-p)^{-1},(1-pqr)(1-p)^{-1}\} \\
& \{-(1-pr)(1-q)^{-1},(1-pqr)(1-q)^{-1}\} \\
& \{-(1-pq)(1-r)^{-1},(1-pqr)(1-r)^{-1}\} = 1 \\
(S7) & \{u_1,1+qu_1\}\{u_2(1-qu_1)^{-1},[1+q(u_1+u_2)](1+qu_1)^{-1}\} \\
& = \{v_1,1+qv_1\}\{v_2(1-qv_1)^{-1},[1+q(v_1+v_2)](1+qv_1)^{-1}\}
\end{array}$$

for all $p,q,r \in P$, $u,v,u_1,u_2,v_1,v_2 \in R^*$ such that $u_1+u_2 = v_1+v_2 \in P$

We will see that two relations may be omitted by making use of Kolster's more general result for local rings [MK3] as applied to the case of a commutative discrete valuation ring, R. It states that if R is a DVR, then

$$K_2(R) \cong \bar{D}_0(R),$$

where $\bar{D}_0(R)$ is defined to be the abelian group generated by $\{u,v\}$ with $u,v \in R^*$, subject to the relations

(V1) $\{u, 1-u\} = 1$
(V2) $\{uv, w\} = \{v, w\}\{u, w\}$
(V3) $\{u, v\}\{v, u\} = 1$
(V4) $\{u, -u\} = 1$
(V5) $\{u_1, 1+qu_1\}\{(1+u_1q)^{-1}u_2, (1+qu_1)^{-1}[1+q(u_1+u_2)]\}$
$\quad = \{v_1, 1+qv_1\}\{(1+v_1q)^{-1}v_2, (1+qv_1)^{-1}[1+q(v_1+v_2)]\}$,
where $q \in \operatorname{rad} R$, $u_1+u_2 = v_1+v_2 \in \operatorname{rad} R$

Kolster proves that this group, $\bar{D}_0(R)$, is isomorphic to $D_1(R)$, the abelian group generated by Dennis-Stein symbols $\langle a,b\rangle$ with $1+ab \in R^*$, subject to the relations

(R1) $\langle a,b\rangle\langle -b,-a\rangle = 1$
(R2) $\langle ay,b\rangle\langle ba,y\rangle = \langle a,yb\rangle$
(R4′) $\langle a_1+a_2,b\rangle = \langle \theta^{-1}a_2\varepsilon^{-1}, \varepsilon b\varepsilon\rangle\langle a_1,b\rangle$, $\varepsilon = 1+ba$, $\theta = \varepsilon\varepsilon^{-1}$

We have used here that R1-R4 are equivalent to R1-R3,R4′ and that in the commutative case,

$$\text{(R3)} \quad \langle a,b\rangle\langle c,d\rangle\langle a,b\rangle^{-1} = \langle \theta c\theta^{-1}, \theta d\theta^{-1}\rangle, \ \theta = (1+ab)(1+ba)^{-1}$$

is simply a statement of commutativity and is thus unnecessary. (See Chapter III for the more general non-commutative versions of $\bar{D}_0(R)$ and $D_1(R)$.)

It is clear immediately that we have the correspondence

$$\begin{aligned} S1 &= V2 \\ S2 &= V3 \\ S3 &= V4 \\ S4 &= V1 \\ S7 &= V5 \end{aligned}$$

between the relations of Dennis-Stein and Kolster, from which we infer that relations S5,S6 of the Dennis-Stein presentation are consequences of the remaining relations. Thus, we may restate Theorem 4.1 as follows

**(New) Theorem 4.2** If $R$ is a commutative discrete valuation ring with maximal ideal $P = \text{rad } R$, then $K_2(R) \cong S(R)$, the abelian group generated by the Steinberg symbols $\{u,v\}$, $u,v \in R^*$ subject to the five defining relations

(S1) $\{u_1u_2, v\} = \{u_1, v\}\{u_2, v\}$

(S2) $\{u,v\} = \{v,u\}^{-1}$

(S3) $\{u,-u\} = 1$

(S4) $\{u, 1-u\} = 1$, if $1-u \in R^*$

(S7) $\{u_1, 1+qu_1\}\{u_2(1-qu_1)^{-1}, [1+q(u_1+u_2)](1+qu_1)^{-1}\}$
$= \{v_1, 1+qv_1\}\{v_2(1-qv_1)^{-1}, [1+q(v_1+v_2)](1+qv_1)^{-1}\}$

for all $p,q,r \in P$, $u,v,u_1,u_2,v_1,v_2 \in R^*$ such that $u_1+u_2 = v_1+v_2 \in P$.

We now proceed to show explicitly how S5 and S6 are consequences of relations S1-S4,S7. We introduce the map

$$\Psi : \bar{D}_0(R) \longrightarrow D_1(R)$$

defined on generators by

$$\Psi(\langle a,b\rangle) = \{1+ab, b\} \qquad \text{if } b \in R^*$$

$$= \{-a, 1+ba\} \qquad \text{if } a \in R^*$$
$$= \langle\!\langle -(1+a)(1-b)^{-1}, -b \rangle\!\rangle \quad \text{if } a, b \in \operatorname{rad} R$$

which is an isomorphism for commutative discrete valuation rings

We note that $\Psi$ is an isomorphism for any local ring, R, even if R is non-commutative [MK3]. However, we have stated the groups $\bar{D}_0(R)$ and $D_1(R)$ in their commutative form. $\Psi$ is an isomorphism in the non-commutative case only if we apply it to the more general non-commutative versions of $\bar{D}_0(R)$ and $D_1(R)$ which were presented in Chapter III.

To see directly that S5 is a consequence of S1-S4,S7, we will show the following:

Step 1. S1,S2 implies $\Psi$(R2($\geq$2 units)), where R2($\geq$2 units) is the relation R2 under the condition that at least two of a,b,y are units, and where $\Psi$(R2($\geq$2 units)) is its image under $\Psi$.

Step 2. S1-S4,S7 implies $\Psi$(R4').

Step 3. R2($\geq$2 units), R4' implies R2($\geq$2 $\in$ rad), so that $\Psi$(R2($\geq$2 units)), $\Psi$(R4') implies $\Psi$(R2($\geq$2 $\in$ rad)),

where R2($\geq$2 $\in$ rad) is the relation R2 under the condition that at least two of a,b,y are in rad R. Of course, by exhaustion, R2($\geq$2 $\in$ rad) and R2($\geq$2 units) imply R2, and R2($\geq$2 $\in$ rad) implies the relations R2(2 $\in$ rad) and R2(3 $\in$ rad) as special cases

and Step 4. $\Psi$(R2($\geq$2 units)) implies S5

from which we conclude that S5 is a consequence of S1-S4,S7.

We now proceed by proving steps 1-4.

**Step 1**: Rewrite the relation R2 in the form

$$\langle -ay,-b\rangle\langle -ba,-y\rangle\langle -yb,-a\rangle = 1,$$

and assume that at least two of a,b,y are units
Then its image under $\Psi$ is

$$\{ay,1+bay\}\{1+bay,-y\}\{1+yba,-a\}$$

so by (S1),

$$\begin{aligned}\{ay,1+bay\} &= \{-y^{-1}ay,1+bay\}\{-y,1+bay\}\\ &= \{-a,1+yba\}\{-y,1+bay\}\end{aligned}$$

which holds by (S2) Thus

$$S1,S2 \text{ implies } \Psi(R2(\geq 2 \text{ units}))$$

as desired

**Step 2**: If $b \in R^*$, then R4' maps to

$$\{1+(a_1+a_2)b,b\} = \{1+a_1b,b\}\{1+(1+a_1b)^{-1}a_2b,b\},$$

so assume that $b \in \text{rad } R$

**Definition 4.3** The notation $\langle\!\langle,\rangle\!\rangle$ is used to denote a Steinberg symbol as follows

$$\langle\!\langle u,q\rangle\!\rangle = \{u,1+qu\}, \qquad \text{where } u \in R^*,\ q \in \text{rad } R$$

We now break Step 2 into five cases
Case 1 $a_1, a_2, a_1+a_2 \in R^*$
Then R4' maps to

$$\langle\langle -(a_1+a_2),-b\rangle\rangle = \langle\langle -a_1,-b\rangle\rangle\langle\langle -(1+a_1b)^{-1}a_2,-b\rangle\rangle$$

by the following

**Claim 4.4** $\langle\langle u_1+u_2,q\rangle\rangle = \langle\langle u_1,q\rangle\rangle\langle\langle (1+u_1q)^{-1}u_2,q\rangle\rangle$
is a consequence of relations S1-S4, for all $u_1, u_2, u_1+u_2 \in R^*$

**Proof** (of claim) Let $\sigma = 1+u_1q$ Then

$$\begin{aligned}\langle\langle \sigma^{-1}u_2,q\rangle\rangle &= \{\sigma^{-1}u_2, 1+q\sigma^{-1}u_2\}\\ &= \{\sigma^{-1}u_2, 1+\sigma^{-1}qu_2\}\\ &= \{\sigma^{-1}u_2, \sigma^{-1}(1+q(u_1+u_2))\}\\ &= \{\sigma,\sigma^{-1}u_2\}\{\sigma^{-1}u_2, 1+q(u_1+u_2)\}\\ &= \{\sigma,u_1\}\{\sigma,u_1^{-1}\sigma^{-1}u_2\}\{\sigma^{-1}u_2,1+q(u_1+u_2)\}\end{aligned}$$

by

$$\{u,vw\} = \{u,v\}\{u,w\}$$

which is a direct consequence of S1 and S2 By S1, we see that

$$\langle\langle u_1+u_2,q\rangle\rangle = \{u_2^{-1}\sigma(u_1+u_2), 1+q(u_1+u_2)\}\{\sigma^{-1}u_2, 1+q(u_1+u_2)\},$$

so it is left to see

$$\{u_2^{-1}\sigma(u_1+u_2), 1+q(u_1+u_2)\} = \{\sigma, u_1^{-1}\sigma^{-1}u_2\}$$

But,

$$1-u_2^{-1}\sigma(u_1+u_2) = -u_2^{-1}u_1(1+q(u_1+u_2)),$$

so

$$\begin{aligned}\{u_2^{-1}\sigma(u_1+u_2), 1+q(u_1+u_2)\} &= \{u_2^{-1}\sigma(u_1+u_2), -u_1^{-1}u_2\}\\ &= \{u_2^{-1}\sigma u_1u_1^{-1}(u_1+u_2), -u_1^{-1}u_2\}\\ &= \{u_2^{-1}\sigma u_1, -u_1^{-1}u_2\}\\ &= \{-u_1^{-1}u_2, u_1^{-1}\sigma^{-1}u_2\}\end{aligned}$$

We finish proving the claim by noting that

$$\{-u_1^{-1}u_2, u_1^{-1}\sigma^{-1}u_2\} = \{-u_2u_1^{-1}, \sigma^{-1}\}$$
$$= \{\sigma, -u_2u_1^{-1}\}$$
$$= \{\sigma, \sigma^{-1}u_2u_1^{-1}\}$$

$\square$

Case 2 $a_1, a_2 \in R^*$, $a_1+a_2 \in \text{rad } R$

Because $\{1, u\} = 1$ and $1+v_1q = v_2$, we have

$$\langle\langle (1+v_1q)^{-1}v_2, q\rangle\rangle = 1,$$

where we have chosen $v_1 = (u_1+u_2-1)(1+q)^{-1}$, $v_2 = (1+(u_1+u_2)q)(1+q)^{-1}$, so that $v_1, v_2 \in R^*$ and $v_1+v_2 = u_1+u_2$

Then, by S7, we may conclude that

$$\langle\langle u_1, q\rangle\rangle\langle\langle (1+u_1q)^{-1}u_2, q\rangle\rangle = \langle\langle (u_1+u_2-1)(1+q)^{-1}, q\rangle\rangle,$$

if $q$, $u_1+u_2 \in \text{rad } R$ From this, case 2 follows

Case 3 $a_1 \in R^*$, $a_2 \in \text{rad } R$

We want

$$\langle\langle -(a_1+a_2), -b\rangle\rangle = \langle\langle -a_1, -b\rangle\rangle\langle\langle -(1+(1+a_1b)^{-1}a_2)(1-b)^{-1}, -b\rangle\rangle$$

Let $u_1 = -a_1$, $u_2 = -(1+a_1b+a_2)(1-b)^{-1}$, $v_1 = -(1-b)^{-1}$, $v_2 = -(a_1+a_2)(1-b)^{-1}$ Then, using that $1+(1+a_1b)^{-1}a_2 = (1+a_1b)^{-1}(1+a_1b+a_2)$ and $u_1+u_2 = v_1+v_2 = -(1+a_1+a_2)(1-b)^{-1}$, we get by applying first S7 and then S3 to the right hand side,

$$\langle\langle -a_1, -b\rangle\rangle\langle\langle -(1+(1+a_1b)^{-1}a_2)(1-b)^{-1}, -b\rangle\rangle$$
$$= \langle\langle -(1-b)^{-1}, -b\rangle\rangle\langle\langle -(1-b)(a_1+a_2)(1-b)^{-1}, -b\rangle\rangle$$
$$= \langle\langle -(a_1+a_2), -b\rangle\rangle$$

Case 4 $a_1 \in \text{rad } R$, $a_2 \in R^*$

We have to show

$$\langle\langle -(a_1+a_2),-b\rangle\rangle = \langle\langle -(1+a_1)(1-b)^{-1},-b\rangle\rangle\langle\langle -(1+a_1b)^{-1}a_2,-b\rangle\rangle$$

Let $u_1 = -(1+a_1)(1-b)^{-1}$, $u_2 = -a_2(1-b)^{-1}$ then $u_1+u_2 = -(1+a_1+a_2)(1-b)^{-1}$ and the right hand side equals

$$\langle\langle u_1,-b\rangle\rangle\langle\langle 1-u_1b)^{-1}u_2,-b\rangle\rangle,$$

and case 4 follows as in case 3

Case 5 $a_1,a_2 \in \text{rad } R$
We have to show

$$\begin{aligned}&\langle\langle -(1+a_1+a_2)(1-b)^{-1},-b\rangle\rangle\\ &\quad = \langle\langle -(1+a_1)(1-b)^{-1},-b\rangle\rangle\langle\langle -(1+a_1b)^{-1}(1+a_1b+a_2)(1-b)^{-1},-b\rangle\rangle\end{aligned}$$

Let $u_1 = -(1+a_1)(1-b)^{-1}$, $u_2 = -(1+a_1b+a_2)(1-b)^{-2}$, $v_1 = -(1-b)^{-1}$, $v_2 = -(1+a_1+a_2)(1-b)^{-2}$, and we are done by applying S7 again

Thus, R4' is a consequence of relations S1-S4,S7
We now continue with our four main steps
**Step 3**: By R4', we see that

$$\begin{aligned}\langle -ay,-b\rangle &= \langle -a,-b\rangle\langle (1+ab)^{-1}a(1-y),-b\rangle\\ \langle -ba,-y\rangle &= \langle -ba,(1+ab)^{-1}(1-y)\rangle\langle -ba,-1\rangle\\ \langle -yb,-a\rangle &= \langle -b,-a\rangle\langle (1+ab)^{-1}(1-y)b,-a\rangle\end{aligned}$$

Let $u = -(1+ab)^{-1}(1-y)$. Then $u \in R^*$ and by R3 (which is a consequence of V6) we can rewrite the relation R2($\geq 2 \in$ rad) as

$$\langle -au,-b\rangle\langle -ba,-u\rangle\langle -ub,-a\rangle\langle -a,-b\rangle\langle -ba,-1\rangle\langle -b,-a\rangle = 1.$$

Thus we have a product of two relations of type R2($\geq$2 units), so we have seen that R2($\geq$2 units), R4$'$ implies R2($\geq 2 \in$ rad) in the presence of S1-S4,S7, and so

$$\Psi(\text{R2}(\geq 2 \text{ units})),\ \Psi(\text{R4}')\ \text{implies}\ \Psi(\text{R2}(\geq 2 \in \text{rad})),$$

as desired

**Step 4**: We conclude by showing that $\Psi(\text{R2}(\geq 2 \in \text{rad}))$ implies S5. Let $a \in R$, and $b, y \in P = \text{rad } R$. Then the relation R2 may be expressed as

$$1 = \langle -ay, -b \rangle \langle -ba, -y \rangle \langle -yb, -a \rangle,$$

which maps under $\Psi$ to

$$\begin{aligned} 1 = {} & \{-(1-ay)(1+b)^{-1}, 1-(1+b)^{-1}(1-ay)b\} \\ & \{-(1-ba)(1+y)^{-1}, 1-(1+y)^{-1}(1-ba)y\}\{1+yba, -a\} \end{aligned}$$

i.e.

$$\begin{aligned} \{-a, 1+yba\} = {} & \{-(1-ay)(1+b)^{-1}, (1+b)^{-1}(1+bay)\} \\ & \{-(1-ba)(1+y)^{-1}, (1+y)^{-1}(1+yba)\} \end{aligned}$$

so with $v=-1$, $p=-b$, $q=-y$, we get

$$\begin{aligned} \{v, 1-pqv\} = {} & \{-(1-qv)(1-p)^{-1}, (1-pqv)(1-p)^{-1}\} \\ & \{-(1-pv)(1-q)^{-1}, (1-pqv)(1-q)^{-1}\} \end{aligned}$$

which is just S5.

This shows directly that S5 is a consequence of S1-S4,S7. We see that S6 is a consequence of S1-S4,S7 by noting that $\Psi(\text{R2}(3 \in \text{rad}))$ implies S6.
Let $a, b, y \in P$. Then R2($3 \in$ rad) maps under $\Psi$ to

$$1 = \langle\langle -(1-ay)(1+b)^{-1}, b\rangle\rangle\langle\langle -(1-ba)(1+y)^{-1}, y\rangle\rangle\langle\langle -(1-yb)(1+a)^{-1}, a\rangle\rangle$$

i.e

$$\begin{aligned} 1 = &\{-(1-ay)(1+b)^{-1}, 1-(1+b)^{-1}(1-ay)b\} \\ &\{-(1-ba)(1+y)^{-1}, 1-(1+y)^{-1}(1-ba)y\} \\ &\{-(1-yb)(1+a)^{-1}, 1-(1-yb)(1+a)^{-1}a\} \end{aligned}$$

so with $q=-y, r=-a, p=-b$ we get

$$\begin{aligned} \{v, 1-pqv\} = &\{-(1-qv)(1-p)^{-1}, (1-pqv)(1-p)^{-1}\} \\ &\{-(1-pv)(1-q)^{-1}, (1-pqv)(1-q)^{-1}\} \end{aligned}$$

which is S6, as desired

We note that the computations done here concerning the redundancy of the Dennis-Stein relations for a discrete valuation ring are not original, and also plays a role in Kolster's determination of $K_2(R)$ for not-necessarily commutative local rings

## CHAPTER V: On a Normal Form for Semilocal Rings

In Kolster's calculation of $K_2(R)$ for local rings (not necessarily commutative), the normal form

$$x = \rho_2 h(u_2) w(v) x_2(c) x_1(d)$$

where $\rho_2$ is in the image of $K_2(2,R)$ in $St_1'(2,R)$, $u_2 \in R^*$, $c, d, v \in R$, and we use the convention $w(0) := 1$

The discovery of a normal form for elements of the Steinberg group is a crucial step in the calculation of $K_2(R)$, so it is natural to ask if the above normal form holds in the more general case of semilocal rings. This would be a positive step towards the computation of $K_2(R)$ for semilocal rings in terms of the standard Dennis-Stein symbols.

We show by counterexample, that Kolster's presentation for the local case does not apply also the the commutative semilocal case. We begin by noting that any element $x \in St(2,R)$, where R is commutative semilocal, may be written in the normal form

$$x = \rho_1 h(u_1) x_2(t) x_1(a) x_2(b)$$

where $\rho_1 \in K_2(2,R)$, $u \in R^*$, $a, b, t \in R$ [MK2]. Thus, the same presentation holds for the quotient group, $St_1'(2,R)$.

Suppose any element $x \in St(2,R)$ could be written in the desired normal form

$$x = \rho_2 h(u_2) w(v) x_2(c) x_1(d)$$

where $\rho_2$ is in the image of $K_2(2,R)$ in $St_1'(2,R)$, $u_2 \in R^*$, $c, d, v \in R$, and we use the convention $w(0) := 1$. Then there would be solutions to the corresponding matrix equations under the Steinberg map

On the one hand, we have the known normal form

$$\rho_1 h(u_1)x_2(t)x_1(a)x_2(b)$$

which represents all elements of the modified Steinberg group, and has the image

$$\begin{pmatrix} u_1(1+ab) & a \\ u_1^{-1}(b+t(1+ab)) & 1+at \end{pmatrix}$$

at matrix level. On the other hand, we have the alternative normal form

$$\rho_2 h(u_2)w(v)x_2(c)x_1(d)$$

which has the image

$$\begin{pmatrix} u_2cv & u_2(1+cd)v \\ -u_2^{-1}v^{-1} & -u_2^{-1}dv^{-1} \end{pmatrix} \qquad \text{if } v \neq 0$$

$$\text{or} \qquad \begin{pmatrix} u_2 & u_2d \\ u_2^{-1}c & u_2^{-1}(1+cd) \end{pmatrix} \qquad \text{if } v = 0$$

We now see that $\rho_2 h(u_2)w(v)x_2(c)x_1(d)$ cannot be a normal form for the commutative semilocal ring $\mathbf{Z}/6\mathbf{Z}$ because there are no values for $u_2, c, d, v$ in $\mathbf{Z}/6\mathbf{Z}$ such that

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} u_2cv & u_2(1+cd)v \\ -u_2^{-1}v^{-1} & -u_2^{-1}dv^{-1} \end{pmatrix} \qquad \text{if } v \neq 0$$

$$\text{or} \qquad \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} u_2 & u_2 d \\ u_2^{-1} c & u_2^{-1}(1+cd) \end{pmatrix} \qquad \text{if } v = 0.$$

In the first case ($v \neq 0$), we have $3 = -u_2 v$ and $2 = (-u_2 v)$, so $2 = 3d$. But $3d \equiv 0$ or $3 \pmod 6$, so this is impossible.
For the second case ($v=0$), we have $2 = u_2$ and $3 = u_2 c$, so $3 = 2c$. But $2c \equiv 0, 2,$ or $4 \pmod 6$, so this is impossible.

(In fact, by computer analysis, we find that there are 64 examples of coefficients a,b,t,u for which the corresponding matrix level equations are unsolvable over $\mathbf{Z}/6\mathbf{Z}$.)

Thus, we conclude that Kolster's normal form for local rings does not apply more generally to the (even commutative) semilocal case.

## CHAPTER VI: On the Relative $K_2$ of Non-Commutative Rings

In this chapter, we present our main theorem, which is an extension of the Maazen-Stienstra presentation of $K_2(R,I)$ of a commutative ring relative to a split radical pair. We retain the assumption that the map $R \longrightarrow R/I$ splits, but now allow the ring, R, to be non-commutative. The relative ease with which elements of the Steinberg group may be manipulated in the commutative case has forced most computations of $K_2$ to be done first in the commutative case. The determination of $K_2$ depends heavily on finding enough relations in the Steinberg group to be able to first find a normal form for the elements of interest in the Steinberg group, and then finding enough relations among the generators of $K_2$ to prove the desired result. It is these relations which necessarily become more intricate in the non-commutative case. Often, as we saw even in the commutative case of the Dennis-Stein presentation of a discrete valuation ring, the list of relations among generators may be redundant, but later lead to a refined result. (We saw, in Chapter IV, using some of the computational methods of Kolster [MK3], that two of the Dennis-Stein relations are redundant.)

Here, as in most extensions to the non-commutative case, the complexity of computations grows enormously. But, in our proof of the main theorem, we have been able to confine the necessarily extensive computations to a role which has very little effect on the structure of the proof. This role of the computations is one of the factors which lead us to believe that our assumption that R is local may not be crucial. In fact, we expect that our proof will naturally lead to a determination of the relative $K_2$ of any ring relative to any radical ideal.

One very pleasing effect we gain by limiting computations to a non-structural role in the proof, is that in the midst of an increase in computational complexity, the proof naturally reveals in the normal form, a more explicit view of the structure of elements in the relative $K_2$, possibly facilitating other results concerning the relative and absolute $K_2$ of non-commutative (not necessarily commutative) rings.

The idea of our proof is to find a normal form for elements in

$$\ker[St'(2,R) \longrightarrow St'(2,R/I)]$$

and to prove that every element in this kernel can be represented uniquely by the normal form. This is done by creating a group, $\mathfrak{M}_\triangle$, whose image in $St'(2,R)$ is isomorphic to this kernel, and which consists only of elements expressible in the normal form.

After proving that this normal form holds, we form a direct product $St'(2,R/I)\mathfrak{M}_\triangle$, where $St'(2,R/I)$ acts on $\mathfrak{M}_\triangle$ by conjugation. By explicitly demonstrating an isomorphism

$$\chi\colon St'(2,R/I)\mathfrak{M}_\triangle \longrightarrow St'(2,R)$$

and its inverse, $\Psi$, we can then restrict $\Psi$ to

$$\ker[St'(2,R) \longrightarrow St'(2,R/I)]$$

to get an isomorphism

$$\ker[St'(2,R) \longrightarrow St'(2,R/I)] \cong \mathfrak{M}_\triangle .$$

By uniqueness of the normal form, and a stability theorem due to Kolster, we see that this implies the exactness of

$$1 \longrightarrow K_2(R,I) \longrightarrow D^{+}(R,I) \longrightarrow [R^*,1+I] \longrightarrow 1,$$

where $D^{*}(R,I)$ is a group generated by Dennis-Stein symbols and subject to the 18 relations listed in the statement of the theorem.

**Theorem 6.1.** Let R be a local ring (associative with 1), <u>not necessarily commutative</u>. Let (R,I) be a split pair, i.e. let I be a two-sided proper ideal in R, and let the canonical projection $R \longrightarrow R/I$ split. Then the following sequence is exact:

$$1 \longrightarrow K_2(R,I) \longrightarrow D^{*}(R,I) \longrightarrow [R^{*}, 1+I] \longrightarrow 1,$$

where $D^{*}(R,I)$ is the group generated by the symbols $\langle a,b\rangle$ with $a \in I$ or $b \in I$, and subject to the defining relations:

(1) $\langle a,b\rangle\langle -b,-a\rangle = 1$

(2) $\langle ay,b\rangle\langle ba,y\rangle = \langle a,yb\rangle$

(3) $\langle a,b\rangle\langle c,d\rangle\langle a,b\rangle^{-1} = \langle \theta c\theta^{-1}, \theta d\theta^{-1}\rangle$

(4) $\langle a_1+a_2,b\rangle = \langle \theta^{-1}a_2\hat{\epsilon}^{-1}, \ell b\hat{\epsilon}^{-1}\rangle\langle a_1,b\rangle$

(5) $\langle a+b,c\rangle = \langle b,c\rangle\langle \theta_2^{-1}a\ell_2^{-1}, \ell_2 c\ell_2^{-1}\rangle\{\ell_3\ell_2\hat{\epsilon}_3^{-1}, \hat{\epsilon}_3^{-1}\}\{\ell_2,\ell_3\}$

(6) $\langle a,b+c\rangle = \langle a,b\rangle\langle \ell_5\epsilon_5^{-1}a, c\ell_5^{-1}\rangle\{\ell_6\ell_5\hat{\epsilon}_6^{-1}, \hat{\epsilon}_6^{-1}\}\{\ell_5,\ell_6\}$

(7) $1 = \langle a,b\rangle\{\ell_5,u\}\langle bu(u\ell_5)^{-1}, u\ell_5u^{-1}\epsilon_5^{-1}a\ell_5\rangle\{\theta_9^{-1}\ell_9, \ell_9^{-1}\}\{-\epsilon_9^{-1}, -u\ell_5\}$
$\{-u\ell_5\epsilon_9^{-1}, -\ell_9\}\{-\ell_8u\ell_5\epsilon_9^{-1}\ell_8, -\ell_8^{-1}\}\{-u\ell_5\epsilon_9^{-1}, -1\}$

(8) $\langle a,b\rangle\langle \theta_5^{-1}c, d\theta_5\rangle = \langle c,d\rangle\langle \ell_{11}a, b\ell_{11}^{-1}\rangle\{\ell_{30}\ell_{11}\hat{\epsilon}_{30}^{-1}, \hat{\epsilon}_{30}^{-1}\}$

(9) $\{-uv,-v^{-1}\}\{-v^{-1}u,z\} = \langle -u^{-1}vz(z^{-1}+1)u^{-1}, u(vz)^{-1}u\rangle\{\epsilon_{14},\ell_{14}^{-1}\}$
$\{-\epsilon_{14}^{-1},-u\}\{-u\epsilon_{14}^{-1},-\ell_{14}\}\{\ell_{13}u\epsilon_{14}^{-1}z^{-1}v, (z^{-1}v)^{-1}\}\{v^{-1}z\ell_{13}u\epsilon_{14}^{-1},-1\}$

(10) $\{-uv,-v^{-1}\}\{-v^{-1}u,-1\}\{-v^{-1}uz,-z^{-1}\} = \langle -u^{-1}v(z-1)u^{-1}, -uv^{-1}u\rangle$
$\{\theta_{16}^{-1}\ell_{16},\ell_{16}^{-1}\}\{-\epsilon_{16}^{-1},-u\}\{-u\epsilon_{16}^{-1},-\ell_{16}\}\{-\ell_{15}u\epsilon_{16}^{-1}zv, -(zv)^{-1}\}$

(11) $\{u,v\}\{-vuz,-z^{-1}\} = \langle u^{-1}(1+v)u^{-1}, -uv^{-1}u\rangle\{\theta_{19}^{-1}\ell_{19},\ell_{19}^{-1}\}\{-\epsilon_{19}^{-1},-u\}$

$$\{-u\varepsilon_{19}^{-1},-\ell_{19}\}\langle -u_2^{-1}v^{-1}(z-1)u_2^{-1},-u_2vu_2\rangle\{\theta_{19}^{-1}\ell_{19},\ell_{19}^{-1}\}$$
$$\{-\varepsilon_{19}^{-1},-u_2\}\{-u_2\varepsilon_{19}^{-1},-\ell_{19}\}\{-\ell_{17}u_2\varepsilon_{19}^{-1}zv^{-1},-vz^{-1}\}$$

(12) $$\{u,v\}\{-vuz,-z^{-1}\} = \langle uv(1+z),-z^{-1}vu^{-1}\rangle\{\ell_{13}u\hat{\varepsilon}_{13}^{-1},\hat{\varepsilon}_{13}^{-1}\}\{u,-z^{-1}v\}$$

(13) $$\{-uv,-v^{-1}\}\langle (v^{-1}u)(1+z),-z^{-1}u^{-1}v\rangle\{-z^{-1}v^{-1}uz,-z\}\{-v^{-1}u,z^{-1}\}$$
$$= \langle -u^{-1}v(z-1)u^{-1},-uv^{-1}u\rangle\{\theta_{16}^{-1}\ell_{16},\ell_{16}^{-1}\}\{-\varepsilon_{16}^{-1},-u\}\{-u\varepsilon_{16}^{-1},-\ell_{16}\}$$
$$\{-\ell_{15}u\varepsilon_{16}^{-1}zv,-(zv)^{-1}\}$$

(14) $$\{-uv,-v^{-1}\}\langle -v^{-1}ua,-bu^{-1}v\rangle\{-\ell_5v^{-1}u\hat{\varepsilon}_5^{-1},\hat{\varepsilon}_5^{-1}\}$$
$$= \langle a_2,b_2\rangle\{\theta_{24}^{-1}\ell_{24},\ell_{24}^{-1}\}\{-\varepsilon_{24}^{-1},-u\}\{-u\varepsilon_{24}^{-1},-\ell_{24}\}$$
$$\langle -u_3^{-1}\theta_{20}^{-1}(1-\ell_{20})u_3^{-1},-u_3\ell_{20}^{-1}\theta_{20}u_3\rangle\{\theta_{28}^{-1}\ell_{28},\ell_{28}^{-1}\}$$
$$\{-\varepsilon_{28}^{-1},-u_3\}\{-u_3\varepsilon_{28}^{-1},-\ell_{28}\}\langle u_4^{-1}\varepsilon_{20}^{-1}(-v-1)u_4^{-1},u_4\varepsilon_{20}u_4\rangle$$
$$\{\theta_{32}^{-1}\ell_{32},\ell_{32}^{-1}\}\{-\varepsilon_{32}^{-1},-u_4\}\{-u_4\varepsilon_{32}^{-1},-\ell_{32}\}$$
$$\langle u_6^{-1}v\varepsilon_{20}^{-1}(-\ell_{20}-1)u_6^{-1},u_6\varepsilon_{20}vu_6\rangle\{\theta_{36}^{-1}\ell_{36},\ell_{36}^{-1}\}$$
$$\{-\varepsilon_{36}^{-1},-u_6\}\{-u_6\varepsilon_{36}^{-1},-\ell_{36}\}\{-u_5\ell_5v\varepsilon_{20}^{-1},-(\ell_5v\varepsilon_{20}^{-1})^{-1}\}$$

(15) $$\{u,v\}\langle -(vu)^{-1}a(vu)^{-1},-vubvu\rangle\{\theta_{44}^{-1}\ell_{44},\ell_{44}^{-1}\}\{-\varepsilon_{44}^{-1},-vu\}$$
$$\{-vu\varepsilon_{44}^{-1},-\ell_{44}\}$$
$$= \langle -uv^{-1}av^{-1},-vbvu^{-1}\rangle\{\ell_{43}u\hat{\varepsilon}_{43}^{-1},\hat{\varepsilon}_{43}^{-1}\}\langle u\theta_{43}^{-1}(1-\ell_{43}),\ell_{43}^{-1}\theta_{43}u^{-1}\rangle$$
$$\{\ell_{44}u\hat{\varepsilon}_{44}^{-1},\hat{\varepsilon}_{44}^{-1}\}\langle -u\varepsilon_{43}^{-1}(-v-1),-\varepsilon_{43}u^{-1}\rangle\{\ell_{45}u\hat{\varepsilon}_{45}^{-1},\hat{\varepsilon}_{45}^{-1}\}$$
$$\langle uv\varepsilon_{43}^{-1}(\ell_{43}+1),-\varepsilon_{43}vu^{-1}\rangle\{\ell_{46}u\hat{\varepsilon}_{46}^{-1},\hat{\varepsilon}_{46}^{-1}\}\{u,\ell_5v\varepsilon_{43}^{-1}\}$$

(16) $$\langle uv(z-1),v^{-1}u^{-1}\rangle\{\ell_{15}u\hat{\varepsilon}_{15}^{-1},\hat{\varepsilon}_{15}^{-1}\}\{u,zv\} = \{u,v\}\{vu,z\}$$

(17) $$\{u,-1\}\{uv,-v^{-1}\} = \langle u^{-1}(v-1)u^{-1},u^2\rangle\{\theta_{41}^{-1}uvu^{-1},uv^{-1}u^{-1}\}\{-u^{-1}v^{-1}u,-u\}$$
$$\{-v^{-1}u,-uvu^{-1}\}\{uv,v^{-1}\}$$

(18) $$\langle uau,u^{-1}bu^{-1}\rangle\{u^{-1}\ell u,u\}\{\ell u\ell,\hat{\varepsilon}^{-1}\} = \langle u^{-1}au^{-1},ubu\rangle\{\theta_{47}^{-1}\ell_{47},\ell_{47}^{-1}\}$$
$$\{-\varepsilon_{47}^{-1},-u\}\{-u\varepsilon_{47}^{-1},-\ell_{47}\}\langle -u_7^{-1}\theta_5^{-1}\ell_5(\ell_5^{-1}-1)u_7^{-1},-u_7(\theta_5^{-1}\ell_5)^{-1}u_7\rangle$$
$$\{\theta_{49}^{-1}\ell_{49},\ell_{49}^{-1}\}\{-\varepsilon_{49}^{-1},-u_7\}\{-u_7\varepsilon_{49}^{-1},-\ell_{49}\}$$
$$\langle u_8^{-1}\varepsilon_5^{-1}(\ell_5+1)u_8^{-1},-u_8\varepsilon_5u_8\rangle\{\theta_{51}^{-1}\ell_{51},\ell_{51}^{-1}\}\{-\varepsilon_{51}^{-1},-u_8\}$$
$$\{-u_8\varepsilon_{51}^{-1},-\ell_{51}\}\{u_9\theta_5^{-1},\theta_5\}$$

where

$\varepsilon_2 = 1+(b)(c)$, $\varepsilon_3 = 1+(\varepsilon_2^{-1}a\ell_2^{-1})(\ell_2 c)$,
$\varepsilon_4 = 1 + (a)(b+c)$, $\varepsilon = \varepsilon_5 = 1+(a)(b)$, $\varepsilon_6 = 1+(\varepsilon_5^{-1}a)(c)$
$\varepsilon_8 = 1+(-u\ell_5 bu)(-u^{-1}\varepsilon_5^{-1}au^{-1})$, $\varepsilon_9 = 1+(bu(u\ell_5)^{-1})(u\ell_5 u^{-1}\varepsilon_5^{-1}a\ell_5)$
$\varepsilon_{11} = 1+(c)(d)$, $\varepsilon_{12} = 1+(\theta_5^{-1}c)(d\theta_5)$
$\theta_{13} = [v,z^{-1}]$, $\varepsilon_{14} = 1+(-u^{-1}v(1+z)u^{-1})(u(vz)^{-1}u)$
$\theta_{15} = [v,z]$, $\varepsilon_{16} = 1+(u^{-1}v(z-1)u^{-1})(uv^{-1}u)$
$\theta_{17} = [v^{-1},z]$, $\varepsilon_{19} = 1+(-u^{-1}(1+v)u^{-1})(uv^{-1}u)$,
$\varepsilon_{19} = 1+(u_2^{-1}v^{-1}(z-1)u_2^{-1})(u_2vu_2)$, and $u_2 = \ell_{38}u\varepsilon_{19}^{-1}$
$\varepsilon_{20} = 1+(-v^{-1}av^{-1})(-vbv)$, $\theta_{21} = \theta_{20}^{-1}\hat{\varepsilon}_{20}^{-1}\varepsilon_{20}$, $\theta_{22} = [\varepsilon_{20}^{-1},v]$
$\theta_{23} = [v\varepsilon_{20}^{-1},\ell_{20}]$, $\varepsilon_{24} = 1+(a_2)(b_2)$, $\theta_{25} = \theta_{24}^{-1}\hat{\varepsilon}_{24}^{-1}\varepsilon_{24}$, $\theta_{26} = [\varepsilon_{24}^{-1},u]$
$\theta_{27} = [u\varepsilon_{24}^{-1},\ell_{24}]$, $\varepsilon_{28} = 1+(u_3^{-1}\theta_{20}^{-1}(1-\ell_{20})u_3^{-1})(u_3\ell_{20}^{-1}\theta_{20}u_3)$
$\theta_{29} = \theta_{28}^{-1}\hat{\varepsilon}_{28}^{-1}\varepsilon_{28}$, $\theta_{30} = [\varepsilon_{28}^{-1},u_3]$, $\theta_{31} = [u_3\varepsilon_{28}^{-1},\ell_{28}]$
$\varepsilon_{32} = 1+(u_4^{-1}\varepsilon_{20}^{-1}(-v-1)u_4^{-1})(u_4\varepsilon_{20}u_4)$, $\theta_{33} = \theta_{32}^{-1}\hat{\varepsilon}_{32}^{-1}\varepsilon_{32}$, $\theta_{34} = [\varepsilon_{32}^{-1},u_4]$
$\theta_{35} = [u_4\varepsilon_{32}^{-1},\ell_{32}]$, $\varepsilon_{36} = 1+(u_6^{-1}v\varepsilon_{20}^{-1}(-\ell_{20}-1)u_6^{-1})(u_6\varepsilon_{20}u_6)$
$\theta_{37} = \theta_{36}^{-1}\hat{\varepsilon}_{36}^{-1}\varepsilon_{36}$, $\theta_{38} = [\varepsilon_{36}^{-1},u_6]$, $\theta_{39} = [u_6\varepsilon_{36}^{-1},\ell_{36}]$
$\theta_{40} = [u_5\ell_{11}v\varepsilon_{20}^{-1},(\ell_{11}v\varepsilon_{20}^{-1})^{-1}]$,
$u_3 = \ell_{20}u\varepsilon_{24}^{-1}$, $u_4 = \ell_{21}u_3\varepsilon_{28}^{-1}$, $u_5 = \ell_{23}u_6\varepsilon_{36}^{-1}$, $u_6 = \ell_{22}u_4\varepsilon_{32}^{-1}$
$a_2 = u^{-1}v^{-1}av^{-1}u^{-1}$, $b_2 = uvbvu$
$\theta_{41} = u^{-1}vu^2vu^{-1}$
$\varepsilon_{42} = 1+((vu)^{-1}a(vu)^{-1})(vubvu)$, $\varepsilon_{43} = 1+(v^{-1}av^{-1})(vbv)$,
$\theta_{44} = [\theta_{43}^{-1}\ell_{43},\ell_{43}^{-1}]$, $\theta_{45} = [\varepsilon_{43}^{-1},v]$, $\theta_{46} = [v\varepsilon_{43}^{-1},\ell_{43}]$.
and $\{u,v\}$ means $\langle u(v-1),u^{-1}\rangle$.

Before proving the theorem, we present some comments

In the exact sequence of the theorem, the inclusion map is the obvious map defined on generators as

$$\begin{aligned} K_2(R,I) &\longrightarrow D^*(R,I) \\ \langle a,b\rangle &\longrightarrow \langle a,b\rangle . \end{aligned}$$

The second map is essentially the Steinberg map, and maps the symbol $\langle a,b\rangle$ to the 1,1 entry of its image in the elementary group, namely the element $\theta = \varepsilon\hat{\varepsilon}^{-1}$ The commutator subgroup which is the image of the second map, is related to the correspondence between Dennis-Stein symbols and Steinberg symbols. In particular, in a local ring, we may write the symbol $\langle a,b\rangle$ as a Steinberg symbol as follows:

$$\begin{aligned} \langle a,b\rangle &= \{1+ab,b\}, && \text{if } b\in R^* \\ &= \{-a,1+ba\}, && \text{if } a\in R^* \\ &= \langle\!\langle -(1+a)(1-b)^{-1},-b\rangle\!\rangle, && \text{if } a,b\in \text{Rad } R \end{aligned}$$

Recall that under the Steinberg map, $\{u,v\}$ maps to the matrix with $[u,v]$ in the 1,1 position. Thus, by the correspondence between Dennis-Stein and Steinberg symbols, the 1,1 element of the image of a Dennis-Stein symbol is also the 1,1 element of the image of a Steinberg symbol, which is a commutator. When we require that either a or b is in I, then it is easy to see that this commutator is also in $[R^*,1+I]$. The map is clearly surjective because any product of commutators

$$\prod[a_j,1+i_j] \in [R^*,1+I]$$

has the product

$$\prod\langle a_j i,a_j^{-1}\rangle \in D^+(R,I)$$

as a preimage.

In the case that R is <u>commutative</u>, Theorem 6.1 reduces to the Maazen-Stienstra presentation for $K_2$ of split radical pairs, applied to a local ring. We, of course, use the obvious fact that a proper ideal in a local ring is necessarily a radical ideal. The relations D1-D3 of Maazen-Stienstra's presentation correspond to relations 1-3 in Theorem 6.1, and relation 4 guarantees that the group generated is abelian. The exact sequence becomes a isomorphism because $[R^*,1+I]=1$ in the commutative case. The relations 5-18 obviously hold in $K_2(R,I)$ because they are direct consequences of fundamental identities in $St'(R,I)$. But in the commutative case, these relations are necessarily consequences of relations 1-4, so we have, as expected, the following restatement of Maazen-Stienstra's result, for a commutative local ring:

**Corollary 6.2**: If $(R,I)$ is a split radical pair in a local ring, $R$, then,

$$\delta : D(R,I) \longrightarrow K_2(R,I)$$

is an isomorphism, where $D(R,I)$ is the abelian group defined by

generators: $\langle a,b\rangle$, one for each couple $(a,b) \in R\times I \cup I\times R$ such that $1 + ab \in R^*$

and relations:
(D1) $\langle a,b\rangle\langle -b,-a\rangle = 1$
(D2) $\langle a,b\rangle\langle a,c\rangle = \langle a,b+c+abc\rangle$
(D3) $\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$.

As seen in Chapter III, relations 1-4 are common in the non-commutative case, e.g. in the determination of the absolute $K_2$ of skew fields and non-commutative local rings. One would like to know whether relations 1-4 would also be a defining set of relations for $D^+(R,I)$ in the relative case. It may turn out that not all of the relations 1-18 are necessary to define $D^+(R,I)$. Our theorem determines $D^+(R,I)$, which is a determination of the relative $K_2(R,I)$. This group needs no modification, but we ask the different question of whether a subset of the relations 1-18 defines the same group, $D^+(R,I)$. This type of question is a natural outgrowth of problems concerning the determination of $K_2(R)$ or $K_2(R,I)$ - for example, two of the relations in Dennis-Stein's presentation for $K_2(R)$ of a commutative discrete valuation ring can be removed, but this was not clear until after the more general approach of Kolster, which considered local rings.

There are two clear methods for reducing the set of defining relations for $D^{+}(R,I)$. The most obvious approach would be to take each relation and use relations 1-4 to show it is a consequence of the first four relations. This may be possible with a very clever and elaborate set of computations.

The other approach would be to use the well-known translational technique of Matsumoto which is used to show a set of relations among symbols is a defining set of relations. One would attempt to define a group of left translations, L, that would the definition of a natural map

$$D^{+}(R,I) \longrightarrow St'(2,R) \longrightarrow L$$

with the goal of proving the injectivity of the composition. In the relative situation, this approach may be feasible if one can determine more precisely how the relative group $D^{+}(R,I)$ is embedded in $St'(2,R)$, or find a simple presentation for the relative Steinberg group $St'(2,R,I)$.

In the more general setting of semilocal rings, this may be an easier question. The first step is to generalize from the commutative to the non-commutative case. We do this in our Theorem 6.1. Using the methods and computations employed in the proof of Theorem 6.1, it seems feasible that a similar result be proved next for semilocal rings. The same stability theorems which allow us to work in $St'(2,R)$ for local rings, allow the semilocal case to be examined in terms of $St(3,R)$, which is still manageable in terms of the identities involved, and has additional advantage of being an unmodified Steinberg group. For the most part, our proof only uses the assumption that R is local to allow us to work in $St'(2,R)$. The generalization from local rings to semilocal rings appears suitable for providing more information about the relations defining $D^{+}(R,I)$. Similarly the generalization from commutative discrete valuation rings to local rings that allowed us in

Chapter IV, to reduce the list of relations given by Dennis-Stein for a commutative discrete valuation ring.

In the important special case, I = Rad R, we may reduce the set of relations to the first four, i.e. show that relations 5-18 are consequences of relations 1-4, thus yielding a determination for the relative $K_2(R, \text{Rad } R)$ of any (not necessarily commutative) split local ring.

**Corollary 6.3**: Let R be a local ring (associative with 1), not necessarily commutative. Let (R, rad R) be a split pair. Then the following sequence is exact.

$$1 \longrightarrow K_2(R, I) \longrightarrow D_1(R) \longrightarrow [R^*, 1+\text{Rad } R] \longrightarrow 1$$

**Proof** We first observe that in the case I=Rad R, the groups $D_1(R)$ and $D^+(R,I)$ have identical sets of generators because in a local ring, an element 1+ab is a unit exactly when either a or b is in the maximal ideal, i.e. the radical. Furthermore, we note that the first four relations in $D^+(R, \text{Rad } R)$ are identical to the defining relations R1-R4 of $D_1(R)$. Finally, we conclude that because relations 5-18 hold also in $D_1(R)$, they are consequences of relations 1-4 by Kolster [MK3]. Thus, $D^+(R, \text{Rad } R) \cong D_1(R)$, and we have the exact sequence

$$1 \longrightarrow K_2(R, \text{Rad } R) \longrightarrow D_1(R) \longrightarrow [R^*, 1+\text{Rad } R] \longrightarrow 1$$

as desired.

□

In the interest of possible generalizations of this theorem, we note which of the hypotheses seem to be essential. Most apparent is the dependence of the theorem upon the assumption that I is a radical ideal (in Theorem 6.1 this is a consequence of the assumption that R is a local ring.) For our proof, it is crucial that I be a radical ideal in order to write every element of $St'(2,R)$ in the normal form. As we have mentioned, the assumption that $R$ is local does not seem to be crucial, except in the choice of the commutator subgroup $[R^*, 1+I]$, which may no longer be applicable in generalizations of the short exact sequence. In fact, as we mentioned briefly above, this commutator subgroup fits neatly into the short exact sequence because of the correspondence between Dennis-Stein and Steinberg symbols, which may not hold for more general classes of rings. One reasonable alternative would be to use the normal subgroup generated by the elements $\theta = (1+ab)(1+ba)^{-1} \in R^*$, $a,b \in I$. The assumption that $R \longrightarrow R/I$ splits seems to be non-essential, and a proof along the lines of Keune [FK2] should suffice to see the non-split case.

Before proving the theorem, we prove some identities in $St'(2,R)$.

**Proposition 6.4** The following identities hold in $St'(2,R)$.

(P1) $x_1(0) = 1$

(P2) $x_1(a)x_1(b) = x_1(a+b)$

(P3) $x_1(a)w(u) = w(u)x_2(-u^{-1}au^{-1})$

(P4) $x_2(a)w(u) = w(u)x_1(-uau)$

(P5) $x_1(a)x_2(b) = \langle a,b\rangle w(\varepsilon)w(-1)x_2(\varepsilon b)x_1(\varepsilon^{-1}a)$

(P6) $h(u)h(v) = \{u,v\}h(vu)$

(P7) $h(u)^{-1} = \{u,u^{-1}\}h(u^{-1})$

(P8) $w(1)w(-1) = 1$

(P9) $w(u)w(-u) = 1$

(P10) $w(u)w(v) = \{-uv,-v^{-1}\}w(-v^{-1}u)w(-1)$

(P11) $\{u,-u\} = 1$

(P12) $w(1)w(u)w(1) = \{-u^{-1},-1\}w(-u^{-1})$

(P13) $w(u)w(-1)w(v) = \{u,v\}w(vu)$

(P14) $x_1(c)\langle a,b\rangle = \langle a,b\rangle x_1(\theta^{-1}c)$

(P15) $x_2(c)\langle a,b\rangle = \langle a,b\rangle x_2(c\theta)$

(P16) $w(u)w(-1)\langle a,b\rangle = \langle uau,u^{-1}bu^{-1}\rangle\{u^{-1}\varepsilon u,u\}\{\varepsilon u\varepsilon,\hat{\varepsilon}^{-1}\}w(u)w(-1)$,
where $\varepsilon=1+ba$

(P17) $w(u)\langle a,b\rangle = \langle -u^{-1}au^{-1},-ubu\rangle\{\theta_0^{-1}\varepsilon_0,\hat{\varepsilon}_0^{-1}\}\{-\varepsilon_0^{-1},-u\}\{-u\varepsilon_0^{-1},-\varepsilon\}w(\varepsilon u\varepsilon_0^{-1})$,

where $\epsilon = 1 + ab$, $\hat{\epsilon} = 1 + ba$, $\theta = \epsilon\hat{\epsilon}^{-1}$,
$\epsilon_0 = 1 + (u^{-1}au^{-1})(ubu)$, $\hat{\epsilon}_0 = 1 + (ubu)(u^{-1}au^{-1})$,
and $\theta_0 = \epsilon_0\hat{\epsilon}_0^{-1}$.

We have abbreviated the index 1,2 by just 1 (similarly 2,1 represents 2). We have also used the index variable i, which may take on the value of 1 or 2, i.e. 1,2 or 2,1.

**Proof**

See [MK2] for a proof of the identities. Proofs of original relations which are not in [MK2], follow (See also Theorem 1.10).

First P5. We note that

$$\langle a,b\rangle = x_1(a)x_2(b)x_1(-\epsilon^{-1}a)x_2(-\hat{\epsilon}b)h(\hat{\epsilon})^{-1}$$

so we have

$$\langle a,b\rangle h(\hat{\epsilon})x_2(\hat{\epsilon}b)x_1(\epsilon^{-1}a) = x_1(a)x_2(b)$$

i.e. $$\langle a,b\rangle w(\hat{\epsilon})w(-1)x_2(\hat{\epsilon}b)x_1(\epsilon^{-1}a) = x_1(a)x_2(b)$$

i.e. $$x_1(a)x_2(b) = \langle a,b\rangle w(\hat{\epsilon})w(-1)x_2(\hat{\epsilon}b)x_1(\epsilon^{-1}a).$$

which is the identity desired.

Next we prove P6.

Note that

$$\{u,v\} = h(uv)h(u)^{-1}h(v)^{-1}$$

so

$$\{u,v\}h(v)h(u) = h(uv)$$

$$h(v)h(u) = \{u,v\}^{-1}h(uv)$$

i.e. $$h(u)h(v) = \{u,v\}h(vu)$$

as desired

As for P7, note that by definition of Steinberg symbols,

$$\{u,v\} = h(uv)h(u)^{-1}h(v)^{-1}$$

so

$$\begin{aligned}\{u,u^{-1}\} &= h(1)h(u)^{-1}h(u^{-1})^{-1} \\ &= h(u)^{-1}h(u^{-1})^{-1}\end{aligned}$$

Thus,

$$h(u)^{-1} = \{u,u^{-1}\}h(u^{-1})$$

as desired.

For P10,

$$\begin{aligned}\mathbf{w}(u)\mathbf{w}(v) &= \mathbf{w}(u)\mathbf{w}(-1)\mathbf{w}(1)\mathbf{w}(v) \\ &= h(u)h(-v)^{-1} \\ &= h(u)[\{-v,-v^{-1}\}h(-v^{-1})] \\ &= h(u)h(-v^{-1})\{-v,-v^{-1}\} \\ &= \{u,-v^{-1}\}h(-v^{-1}u)\{-v,-v^{-1}\} \\ &= \{u,-v^{-1}\}\{-v,-v^{-1}\}h(-v^{-1}u) \\ &= {}^{-v}\{v^{-1}uv,-v^{-1}\}\{-v,-v^{-1}\}h(-v^{-1}u) \\ &= \{-vv^{-1}uv,-v^{-1}\}h(-v^{-1}u) \\ &= \{-uv,-v^{-1}\}\mathbf{w}(-v^{-1}u)\mathbf{w}(-1)\end{aligned}$$

$$\begin{aligned}\mathbf{w}(1)\mathbf{w}(u)\mathbf{w}(1) &= \mathbf{w}(1)\mathbf{w}(u)\mathbf{w}(-1)\mathbf{w}(1)\mathbf{w}(1) \\ &= \mathbf{w}(u^{-1})\mathbf{w}(1)\mathbf{w}(1) \\ &= \{-u^{-1},-1\}\mathbf{w}(-u^{-1})\mathbf{w}(-1)\mathbf{w}(1) \\ &= \{-u^{-1},-1\}\mathbf{w}(-u^{-1})\end{aligned}$$

P13.

$$\mathbf{w}(u)\mathbf{w}(-1)\mathbf{w}(v) = h(u)h(v)\mathbf{w}(1)$$

$$
\begin{aligned}
&= \{u,v\}h(vu)\mathbf{w}(1)\\
&= \{u,v\}[h(vu)^{-1}]^{-1}\mathbf{w}(1)\\
&= \{u,v\}[\{vu,(vu)^{-1}\}h((vu)^{-1})]^{-1}\mathbf{w}(1)\\
&= \{u,v\}h((vu)^{-1})^{-1}\{(vu)^{-1},vu\}\mathbf{w}(1)\\
&= \{u,v\}\{(vu)^{-1},vu\}\mathbf{w}(1)\mathbf{w}(-(vu)^{-1})\mathbf{w}(1)\\
&= \{u,v\}\{(vu)^{-1},vu\}\{vu,-1\}\mathbf{w}(vu)\\
&= \{u,v\}\{vu,-1\}\{(vu)^{-1},vu\}\mathbf{w}(vu)\\
&= \{u,v\}\{vu,-1\}\{vu,vu\}\mathbf{w}(vu)\\
&= \{u,v\}\{vu,-1\}^{-1}\{vu,vu\}\mathbf{w}(vu)\\
&= \{u,v\}\{vu,-vu\}\mathbf{w}(vu)\\
&= \{u,v\}\mathbf{w}(vu)
\end{aligned}
$$

P16.

$$
\begin{aligned}
\mathbf{w}(u)\mathbf{w}(-1)\langle a,b\rangle &= h(u)x_1(a)x_2(b)x_1(-\varepsilon^{-1}a)x_2(-\ell b)h(\ell)^{-1}\\
&= x_1(uau)x_2(u^{-1}bu^{-1})x_1(-u\varepsilon^{-1}au)x_2(-u^{-1}\ell bu^{-1})h(u)h(\ell)^{-1}\\
&= \langle uau,u^{-1}bu^{-1}\rangle h(u^{-1}\ell u)h(u)h(\ell)^{-1}\\
&= \langle uau,u^{-1}bu^{-1}\rangle\{u^{-1}\ell u,u\}h(\ell u)h(\ell)^{-1}\\
&= \langle uau,u^{-1}bu^{-1}\rangle\{u^{-1}\ell u,u\}\mathbf{w}(\ell u)\mathbf{w}(-\ell)\\
&= \langle uau,u^{-1}bu^{-1}\rangle\{u^{-1}\ell u,u\}\{\ell u\ell,\ell^{-1}\}\mathbf{w}(u)\mathbf{w}(-1)
\end{aligned}
$$

And finally, P17:

$$
\begin{aligned}
\mathbf{w}(u)\langle a,b\rangle &= \mathbf{w}(u)x_1(a)x_2(b)x_1(-\varepsilon^{-1}a)x_2(-\ell b)h_1(\ell)^{-1}\\
&= x_2(-u^{-1}au^{-1})x_1(-ubu)x_2(u^{-1}\varepsilon^{-1}au^{-1})x_1(u\ell bu)\\
&\qquad h_2(\ell_0)^{-1}h_2(\ell_0)\mathbf{w}(u)h_1(\ell)^{-1}\\
&= x_2(a_0)x_1(b_0)x_2(\varepsilon_0^{-1}a_0)x_1(-\ell_0 b_0)h_2(\ell_0)^{-1}h_2(\ell_0)\mathbf{w}(u)h_1(\ell)^{-1}\\
&= \langle a_0,b_0\rangle_2 h_2(\ell_0)\mathbf{w}(u)h_1(\ell)^{-1}\\
&= \langle a_0,b_0\rangle h(\theta_0^{-1})h(\hat{\varepsilon}_0^{-1})\mathbf{w}(u)h(\ell)^{-1}\\
&= \langle a_0,b_0\rangle \mathbf{w}(\theta_0^{-1})\mathbf{w}(-1)\mathbf{w}(1)\mathbf{w}(-\ell_0)\mathbf{w}(u)h(\ell)^{-1}\\
&= \langle a_0,b_0\rangle \mathbf{w}(\theta_0^{-1})\mathbf{w}(-\ell_0)\mathbf{w}(u)h(\ell)^{-1}\\
&= \langle a_0,b_0\rangle \{\theta_0^{-1}\ell_0,\hat{\varepsilon}_0^{-1}\}\mathbf{w}(\hat{\varepsilon}_0^{-1}\theta_0^{-1})\mathbf{w}(-1)\mathbf{w}(u)\mathbf{w}(1)\mathbf{w}(-\ell)\\
&= \langle a_0,b_0\rangle \{\theta_0^{-1}\ell_0,\hat{\varepsilon}_0^{-1}\}\mathbf{w}(\hat{\varepsilon}_0^{-1}\theta_0^{-1})\mathbf{w}(u^{-1})\mathbf{w}(-\ell)\\
&= \langle a_0,b_0\rangle \{\theta_0^{-1}\ell_0,\hat{\varepsilon}_0^{-1}\}\{-\hat{\varepsilon}_0^{-1}\theta_0^{-1},-u\}\mathbf{w}(-u\hat{\varepsilon}_0^{-1}\theta_0^{-1})\mathbf{w}(-1)\mathbf{w}(-\ell)\\
&= \langle a_0,b_0\rangle \{\theta_0^{-1}\ell_0,\hat{\varepsilon}_0^{-1}\}\{-\hat{\varepsilon}_0^{-1}\theta_0^{-1},-u\}h(-u\hat{\varepsilon}_0^{-1}\theta_0^{-1})h(-\ell)\mathbf{w}(1)\\
&= \langle a_0,b_0\rangle \{\theta_0^{-1}\ell_0,\hat{\varepsilon}_0^{-1}\}\{-\hat{\varepsilon}_0^{-1}\theta_0^{-1},-u\}\{-u\hat{\varepsilon}_0^{-1}\theta_0^{-1},-\ell\}h(\ell u\hat{\varepsilon}_0^{-1}\theta_0^{-1})\mathbf{w}(1)
\end{aligned}
$$

$$= \langle a_0, b_0 \rangle \{\theta_0^{-1}\ell_0, \hat{\varepsilon}_0^{-1}\}\{-\hat{\varepsilon}_0^{-1}\theta_0^{-1}, -u\}\{-u\hat{\varepsilon}_0^{-1}\theta_0^{-1}, -\ell\}w(\ell u \hat{\varepsilon}_0^{-1}\theta_0^{-1})$$
$$= \langle a_0, b_0 \rangle \{\theta_0^{-1}\ell_0, \hat{\varepsilon}_0^{-1}\}\{-\varepsilon_0^{-1}, -u\}\{-u\varepsilon_0^{-1}, -\ell\}w(\ell u \varepsilon_0^{-1})$$

where $a_0 = -u^{-1}au^{-1}$, $b_0 = -ubu$, $\varepsilon_0 = u^{-1}\varepsilon u$, $\ell_0 = u\ell u^{-1}$, $\theta_0 = \varepsilon_0\hat{\varepsilon}_0^{-1}$.

$\square$

**Proof**: (Theorem 6.1)

Let $\pi$ be a partial order on $\{1,2\}$. By $W_\pi$, we denote the words in the letters

$$\begin{array}{lll} & X_i(a), & a \in I \text{ if } ``i \notin \pi" \\ & W(u), & u \in 1+I \\ \text{and} & D, & D \in D^*(R,I) \end{array}$$

with the condition that at least one "W" occurs in every word. (Of course, $W(1)W(-1)$ is allowed.)

Define the map $\beta_\pi$: $W_\pi \longrightarrow St'(2,R)$

by

$$\begin{aligned} X_i(a) &\longrightarrow x_i(a) \\ W(u) &\longrightarrow w(u) \\ D &\longrightarrow D. \end{aligned}$$

**Definition 6.5**: Let $A = A_0A_1A_2$ and $A' = A_0A_1'A_2$ be juxtapositions of three words. We say that $A'$ is obtained from $A$ by **replacing** the subword $A_1$ by $A_1'$.

Only the following replacements and their compositions will be allowed:

(1) $X_i(0) \longrightarrow 1$
(2) $D \longrightarrow (\quad)$, if $D = 1$ in $D^*(R,I)$
(3) $D_1D_2 \longrightarrow (D_1D_2)$, where $(D_1D_2)$ is the product in $D^*(R,I)$
(4) $X_i(a)X_i(b) \longrightarrow X_i(a+b)$
(5) $X_1(a)W(u) \longrightarrow W(u)X_2(-u^{-1}au^{-1})$

$X_2(a)W(u) \longrightarrow W(u)X_1(-uau)$

(6) $X_1(a)X_2(b) \longrightarrow \langle a,b\rangle W(\varepsilon)W(-1)X_2(\varepsilon b)X_1(\varepsilon^{-1}a)$

(7) $W(u)W(-u) \longrightarrow 1$

(8) $W(u)W(v) \longrightarrow \{-uv,-v^{-1}\}W(-v^{-1}u)W(-1)$, $v \neq -1$

(9) $W(u)W(-1)W(v) \longrightarrow \{u,v\}W(vu)$

(10) $X_1(c)\langle a,b\rangle \longrightarrow \langle a,b\rangle X_1(\theta^{-1}c)$

$X_2(c)\langle a,b\rangle \longrightarrow \langle a,b\rangle X_2(c\theta)$

(11) $W(u)\langle a,b\rangle \longrightarrow \langle -u^{-1}au^{-1},-ubu\rangle\{\theta_0^{-1}\varepsilon_0,\hat{\varepsilon}_0^{-1}\}\{-\varepsilon_0^{-1},-u\}\{-u\varepsilon_0^{-1},-\varepsilon\}W(\varepsilon u\varepsilon_0^{-1})$

If B can be obtained from A by a replacement (B = A allowed) we will write $A \geq B$. By an earlier proposition, these replacements correspond to relations which hold in $St'(2,R)$, so $\beta_\pi(A) = \beta_\pi(B)$ if $A \geq B$.

By Lemma 6.7, $\geq$ is a partial order relation on $W_\pi$, and for all $A \in W_\pi$, the set $\{ B \in W_\pi \mid A \geq B \}$ is finite.

**Definition 6.6a**: $\mathbf{M}_\pi$ is defined to be the set of minimal elements of $W_\pi$ under the ordering $\geq$. The elements of $M_\pi$ are exactly the words of the form $DW(u)X_2(a)X_1(b)$ or $DW(u)W(-1)X_2(a)X_1(b)$, where $X_2(a)$
(respectively $X_1(b)$) occurs only if $a \neq 0$ (resp. if $b \neq 0$) and D occurs only if $D \neq 1$. (If necessary, we write $W(1)W(-1)$ to insure that a W occurs.)

By Lemma 6.8, minimal elements are unique, i.e. for each $A \in W_\pi$, there exists exactly one $B \in M_\pi$ such that $A \geq B$.

**Definition 6.6b**: $\mathfrak{M}_\pi$ is defined to be the set of elements in $M_\pi$ with two W's. We make $\mathfrak{M}_\pi$ into a group by defining $A*B$ to be the unique minimal element in $\mathfrak{M}_\pi$ determined by AB. The unit element is $1 = W(1)W(-1)$. The element $[DW(u)W(-1)X_2(a)X_1(b)]^{-1}$ is the minimal word corresponding to
$X_1(-b)X_2(-a)W(1)W(-u)D^{-1}$. By applying the elementary replacements we see that $\mathfrak{M}_\pi$ is closed under multiplication and inverses.

It is also clear that the elements $DW(u)W(-1)$ are closed under multiplication and inverses. Therefore, they form a subgroup of $M_\pi$, and we have a homomorphism

$$\beta_\pi : \mathfrak{M}_\pi \longrightarrow St'(2,R).$$

Suppose next that $\pi'$ is another order relation on $\{1,2\}$ such that $\pi' \le \pi$. Then $W_{\pi}'$ is a subset of $W_\pi$. The transformation rules on $W_{\pi}'$ are restrictions of the transformation rules on $W_\pi$. Moreover, if $A \in W_{\pi}'$, $B \in W_\pi$ and $A \ge B$ then $B \in W_{\pi}'$. Thus by Lemma 6.10, $\mathfrak{M}_\triangle$ is a normal subgroup of $\mathfrak{M}_\pi$, where $\triangle$ is the order relation corresponding to the diagonal in $\{1,2\}\times\{1,2\}$.

The homomorphism $\beta_\triangle$ maps $\mathfrak{M}_\triangle$ to a subgroup of $St'(2,R)$ which contains the elements $X_1(a)$ with $a \in I$. If $A \in \mathfrak{M}_\triangle$, and $a \in R$, one can lift

$$x_1(a)\beta_\triangle(A)x_1(a)^{-1}$$

to

$$X_1(a)*A*X_1(-a)$$

in some $\mathfrak{M}_\pi$. By normality, the lifted expression is in $\mathfrak{M}_\triangle$, so

$$x_1(a)\beta_\triangle(A)x_1(a)^{-1}$$

is in the image of $\mathfrak{M}_\triangle$. Thus the image of $\mathfrak{M}_\triangle$ is a normal subgroup of $St'(2,R)$ which contains all $x_1(a)$ with $a \in I$. Moreover,

$$\beta_\triangle \mathfrak{M}_\triangle \subset \ker[St'(2,R) \longrightarrow St'(2,R/I)].$$

The map

$$St'(2,R/I) \to St'(2,R)/\beta\triangle\mathfrak{M}\triangle$$

defined on generators by

$$x_{ij}(\bar{a}) \to \overline{x_{ij}(a)}$$

(where $\bar{a} \in R/I$, and a is a representative for $\bar{a}$) is a homomorphism, so

$$\beta\triangle\mathfrak{M}\triangle = \ker[St'(2,R) \longrightarrow St'(2,R/I)].$$

Now consider an element of

$$\ker[K_2'(2,R) \longrightarrow K_2'(2,R/I)]$$

It is an element of

$$\ker[St'(2,R) \longrightarrow St'(2,R/I)]$$

and so it is the image of an element

$$DW(u)W(-1)X_2X_1 \in \mathfrak{M}\triangle,$$

i.e. an element

$$Dw(u)w(-1)x_2x_1 \in St'(2,R).$$

But it is also in

$$K_2'(2,R) = \ker[St'(2,R) \longrightarrow E(2,R)],$$

so it maps to **1** under the modified Steinberg map. We have

$$\begin{pmatrix} \pi\theta_1 & \\ & 1 \end{pmatrix} \begin{pmatrix} & u \\ -u^{-1} & \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

so

$$\begin{pmatrix} cu & cub \\ u^{-1}a & u^{-1}(1+ab) \end{pmatrix} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \text{ where } c = \pi\theta_i.$$

Thus, $a=0$, $b=0$, $u=1$, and $c=1$ (i.e. $\prod\theta_i=1$). This conclusion is due to the uniqueness of our normal form.

So far, we have shown that any element of

$$\ker[K_2'(2,R) \longrightarrow K_2'(2,R/I)],$$

is the image under $\beta\triangle$ of an element in $D^+(R,I)$ with $\prod\theta_i=1$. I.e., $\beta\triangle$ maps $\{D^+(R,I) \mid \prod\theta_i=1\}$ surjectively onto

$$\ker[K_2'(2,R) \longrightarrow K_2'(2,R/I)]$$

Next, we exhibit an action of $St'(2,R/I)$ on $\mathfrak{M}\triangle$. For $x \in St'(2,R/I)$ and $A \in \mathfrak{M}\triangle$, the result of the action of $x$ on $A$ is denoted by ${}^{x}A$. For $x_1(s) \in St'(2,R/I)$ and $A \in \mathfrak{M}\triangle$, define

$${}^{x_1(s)}A = X_1(s) * A * X_1(-s),$$

computed in some $\mathfrak{M}_\pi$, with $i \in \pi$. The right hand side is in $\mathfrak{M}\triangle$ by normality of $\mathfrak{M}\triangle$. By Lemma 6.9, this defines an action of $St'(2,R/I)$ on $\mathfrak{M}\triangle$. We also see that $\beta\triangle$ commutes with this action.

Form the semi-direct product $St'(2,R/I)\,\mathfrak{M}\triangle$ according to the action. The elements are pairs $(x,A)$ with $x \in St'(2,R/I)$ and $A \in \mathfrak{M}\triangle$. The multiplication is defined by

$$(x,A)(y,B) = (xy,\ {}^{y^{-1}}A * B).$$

We have a homomorphism

$$\chi\colon St'(2,R/I)\mathfrak{M}\triangle \longrightarrow St'(2,R)$$

defined by

$$(x,A) \longrightarrow x\beta\triangle(A),$$

since

$$(x,A)(y,B) \longrightarrow x\beta\triangle(A)y\beta\triangle(B)$$

and also

$$\begin{aligned}(xy,\ Y^{-1}A*B) \longrightarrow xy(Y^{-1}A*B) &= xy^{-1}\beta\triangle(A)\beta\triangle(B)\\ &= x\beta\triangle(A)y\beta\triangle(B)\end{aligned}$$

We construct an inverse to $\chi$:
Each element of $R$ can be written uniquely as $s+a$ with $s \in R/I$ and $a \in I$. Put

$$\Psi(x_1(s+a)) = (x_1(s), \mathbb{X}_1(a))$$

and extend to all of $St'(2,R)$. We must check to see that the relations in $St'(2,R)$ are satisfied:

$$\begin{aligned}\text{(i)}\quad \Psi(x_1(s+a))\Psi(x_1(t+b)) &= (x_1(s), \mathbb{X}_1(a))(x_1(t), \mathbb{X}_1(b))\\ &= (x_1(s+t), {}^{x_1(t)^{-1}}\mathbb{X}_1(a)*\mathbb{X}_1(b))\\ &= (x_1(s+t), \mathbb{X}_1(a+b))\\ &= \Psi(x_1((s+a)+(t+b)))\end{aligned}$$

$$\begin{aligned}\text{(ii)}\quad \Psi(W(u))\Psi(x_2(s+a))\Psi(W(u))^{-1} &= (w(u),1)(x_2(s), \mathbb{X}_2(a))(w(u)^{-1},1)\\ &= (w(u)x_2(s), {}^{x_2(s)^{-1}}1*\mathbb{X}_2(a))(w(u)^{-1},1)\\ &= (w(u)x_2(s), \mathbb{X}_2(a))(w(u)^{-1},1)\\ &= (w(u)x_2(s)w(u)^{-1}, {}^{w(u)}\mathbb{X}_2(a))\end{aligned}$$

$$= (x_1(-usu), \mathfrak{X}_1(-uau))$$
$$= \Psi(x_1(-u(s+a)u))$$

(iii) and (iv). Note that $\langle a,b\rangle$ is a Steinberg symbol in the Steinberg groups, i.e. a product $h(uv)h(u)^{-1}h(v)^{-1}$. Thus

$$\Psi(\langle a,b\rangle) = \Psi(h(uv)h(u)^{-1}h(v)^{-1}) = (\langle a,b\rangle, 1),$$

and we have:

$$
\begin{aligned}
\text{(iii) } \Psi(\langle a,b\rangle)\Psi(x_1(s+a))\Psi(\langle a,b\rangle^{-1}) &= (\langle a,b\rangle, 1)(x_1(s), \mathfrak{X}_1(a))(\langle a,b\rangle^{-1}, 1) \\
&= (\langle a,b\rangle x_1(s), {}^{x_1(s)^{-1}}1 * \mathfrak{X}_1(a))(\langle a,b\rangle^{-1}, 1) \\
&= (\langle a,b\rangle x_1(s)\langle a,b\rangle^{-1}, {}^{\langle a,b\rangle}\mathfrak{X}_1(a) * 1) \\
&= (x_1(\theta s), \mathfrak{X}_1(\theta a)) \\
&= \Psi(x_1(\theta(s+a))), \quad \text{and similarly for } x_2.
\end{aligned}
$$

$$
\begin{aligned}
\text{(iv) } \Psi(\langle ay,b\rangle)\Psi(\langle ba,y\rangle) &= (\langle ay,b\rangle, 1)(\langle ba,y\rangle, 1) \\
&= (\langle a,yb\rangle, 1) \\
&= \Psi(\langle a,yb\rangle)
\end{aligned}
$$

Thus $\Psi$ satisfies the relations in $St'(2,R)$.

Restricting $\Psi$ to $\ker[St'(2,R) \longrightarrow St'(2,R/I)]$ gives an inverse homomorphism to $\beta_\triangle$. Thus

$$\beta_\triangle : \mathfrak{M}_\triangle \longrightarrow \ker[St'(2,R) \longrightarrow St'(2,R/I)]$$

is an isomorphism. So

$$\ker[K_2'(2,R) \longrightarrow K_2'(2,R/I)]$$

is isomorphic to $\{D^+(R,I) \mid \prod\theta_i = 1\}$, and we get the short exact sequence:

$$1 \longrightarrow \ker[K_2'(2,R) \to K_2'(2,R/I)] \longrightarrow D^{+}(R,I) \longrightarrow [R^*, 1+I] \longrightarrow 1.$$

By the stability theorem of Kolster [MK1], we have

$$K_2'(2,R) \cong K_2(R) \text{ and } K_2'(2,R/I) \cong K_2(R/I).$$

Using also that $K_2(R,I)$ is isomorphic to

$$\ker[K_2(2,R) \to K_2(2,R/I)],$$

we get the desired short exact sequence,

$$1 \longrightarrow K_2(R,I) \longrightarrow D^{+}(R,I) \longrightarrow [R^*, 1+I] \longrightarrow 1$$

This completes our proof of the main theorem, subject to Lemmas 6.7 through 6.10, which follow.

□

**Lemma 6.7** $\geq$ is a partial order relation on $W_\pi$. For all $A \in W_\pi$, the set $\{ B \in W_\pi | A \geq B \}$ is finite.

**Proof**: Consider for a word A, the septuple $r(A) = (r_1, r_2, \ldots, r_7)$ defined by.

$r_1$ = # times an $X_1$ appears before an $X_2$
 **plus** the # times an $X_2$ appears before an $X_2$
$r_2$ = # times an X appears before a W
$r_3$ = # W's
$r_4$ = # times a W appears before a W(u) with $u \neq -1$
$r_5$ = # times a W appears before a D
$r_6$ = # times an X appears before a D
$r_7$ = the total length of the word

Order the tuples lexicographically. By Table 1, we see the effect of any elementary replacement on the tuple. Clearly, $r(B) < r(A)$ if

B can be obtained from A ($B \neq A$) but $r(A) < \infty$, so the proposition follows

$\square$

| | $r_1$ | $r_2$ | $r_3$ | $r_4$ | $r_5$ | $r_6$ | $r_7$ |
|---|---|---|---|---|---|---|---|
| (1) | ~ | ~ | ~ | ~ | ~ | ~ | √ |
| (2) | ~ | ~ | ~ | ~ | ~ | ~ | √ |
| (3) | ~ | ~ | ~ | ~ | ~ | ~ | √ |
| (4) | ~ | ~ | ~ | ~ | ~ | ~ | √ |
| (5) | ~ | √ | x | x | x | x | x |
| (6) | √ | x | x | x | x | x | x |
| (7) | ~ | ~ | √ | x | x | x | x |
| (8) | ~ | ~ | ~ | √* | x | x | x |
| (9) | ~ | ~ | √ | x | x | x | x |
| (10) | ~ | ~ | ~ | ~ | ~ | √ | x |
| (11) | ~ | ~ | ~ | ~ | √ | x | x |

~ = does not increase

√ = decreases

x = do not care

* (8) decreases $r_4$ only if $v \neq -1$, but $v = -1$ is not allowed, so need not be considered.

**Table 1**

**Lemma 6.8** For each $A \in W_\pi$, there exists exactly one $B \in M_\pi$ such that $A \geq B$

**Proof**: We show that the replacement system $W_\pi$ has the Church-Rosser property:

> If A,B,C are words in $W_\pi$ such that $A \geq B$ and $A \geq C$, then there exists a word E such that $B \geq E$ and $C \geq E$.

This suffices to get the proposition. Let A be any word. There exists a word B in $M_\pi$ such that $A \geq B$. If C is another such word, then according to the Church-Rosser property, there will be a word E such that $B \geq E$ and $C \geq E$. By the minimality of B and C, we find $B = E = C$.

Now we prove the Church-Rosser property for $W_\pi$. Let $A,B,C \in W_\pi$ such that $A \geq B$ and $A \geq C$. We look for E such that $B \geq E$ and $C \geq E$. Since the set $\{F \in W_\pi \mid A \leq F\}$ is finite, by an easy induction argument, one can reduce this problem to the case where B and C are obtained from A by elementary replacements. It is obvious that E exists if two distinct subwords are replaced. Since the replacements are now performed within a three letter subword of A (four letter subword if replacement (9) is involved) we may restrict ourselves to the case where A is a three letter word (or a four letter word if replacement (9) is involved.)

The property is trivial if either replacement is a (1),(2),(3), or (7), but there are many other cases to be considered which involve the remaining replacements.

The first case we consider involves just the replacement (4), and we will use the replacement numbers as follows to reference the case.

[[4,4]]

$$(4) \quad X_1(a)X_1(b)X_1(c)$$

$\longrightarrow X_1(a+b)X_1(c)$

$\longrightarrow X_1(a+b+c)$

(4) $X_1(a)X_1(b)X_1(c)$

$\longrightarrow X_1(a)X_1(b+c)$

$\longrightarrow X_1(a+b+c)$

In the notation [(4,4)], used here to describe which case of the Church-Rosser property is being proved, we use the numbers 4,4 to indicate which elementary replacements are being considered. In the letters A,B,C,D,E from our statement of the Church-Rosser property, the first line of each computation is the word A, and we perform the replacement (4) each time. The second lines correspond to the words B and C respectively, which satisfy $A \geq B$ and $A \geq C$. By using elementary replacements, we then show that there is an element E (the bottom line of each part) which satisfies the condition that $B \geq E$ and $C \geq E$, as required.

The remaining cases follow the same general pattern, sometimes also requiring relations from $D^+(R,I)$. The rest of the computations follow:

[(4,5a)]

(4) $X_1(a)X_1(b)W(u)$

$\longrightarrow X_1(a+b)W(u)$

$\longrightarrow W(u)X_2(-u^{-1}(a+b)u^{-1})$

(5a) $X_1(a)X_1(b)W(u)$

$\longrightarrow X_1(a)W(u)X_2(-u^{-1}bu^{-1})$

$\longrightarrow W(u)X_2(-u^{-1}au^{-1})X_2(-u^{-1}bu^{-1})$

$\longrightarrow W(u)X_2(-u^{-1}(a+b)u^{-1})$

[(4,5b)]

(4) $X_2(a)X_2(b)W(u)$

$\longrightarrow X_2(a+b)W(u)$

$$\longrightarrow W(u)\mathbf{X}_1(-u(a+b)u)$$

$$
\begin{aligned}
&(5b) && \mathbf{X}_2(a)\mathbf{X}_2(b)W(u)\\
&\longrightarrow && \mathbf{X}_2(a)W(u)\mathbf{X}_1(-ubu)\\
&\longrightarrow && W(u)\mathbf{X}_1(-u(a+b)u)
\end{aligned}
$$

[[4,6]](i)

$$
\begin{aligned}
&(4) && \mathbf{X}_1(a)\mathbf{X}_1(b)\mathbf{X}_2(c)\\
&\longrightarrow && \mathbf{X}_1(a+b)\mathbf{X}_2(c)\\
&\longrightarrow && \langle a+b,c\rangle W(\ell_1)W(-1)\mathbf{X}_2(\ell_1 c)\mathbf{X}_1(\varepsilon_1^{-1}(a+b)),
\end{aligned}
$$

where $\varepsilon_1 = 1+(a+b)(c)$. Note that when we write a value for $\varepsilon$, we will write the parentheses in such a way as to indicate the values for $\ell$ and $\theta$. For example, here we have $\ell_1 = 1 + (c)(a+b)$ and $\theta_1 = \varepsilon_1\ell_1^{-1}$ as usual.

$$
\begin{aligned}
&(6) && \mathbf{X}_1(a)\mathbf{X}_1(b)\mathbf{X}_2(c)\\
&\longrightarrow && \mathbf{X}_1(a)\langle b,c\rangle W(\ell_2)W(-1)\mathbf{X}_2(\ell_2 c)\mathbf{X}_1\\
&\longrightarrow && \langle b,c\rangle \mathbf{X}_1(\theta_2^{-1}a)W(\ell_2)W(-1)\mathbf{X}_2(\ell_2 c)\mathbf{X}_1\\
&\longrightarrow && \langle b,c\rangle W(\ell_2)\mathbf{X}_2(-\varepsilon_2^{-1}a\ell_2^{-1})W(-1)\mathbf{X}_2(\ell_2 c)\mathbf{X}_1\\
&\longrightarrow && \langle b,c\rangle W(\ell_2)W(-1)\mathbf{X}_1(\varepsilon_2^{-1}a\ell_2^{-1})\mathbf{X}_2(\ell_2 c)\mathbf{X}_1\\
&\longrightarrow && \langle b,c\rangle W(\ell_2)W(-1)\langle \varepsilon_2^{-1}a\ell_2^{-1},\ell_2 c\rangle W(\ell_3)W(-1)\mathbf{X}_2\mathbf{X}_1\\
&\longrightarrow && \langle b,c\rangle W(\ell_2)W(-1)\langle \varepsilon_2^{-1}a\ell_2^{-1},\ell_2 c\rangle W(\ell_3)W(-1)\mathbf{X}_2\mathbf{X}_1\\
&\longrightarrow && \langle b,c\rangle \langle \theta_2^{-1}a\ell_2^{-1},\ell_2 c\ell_2^{-1}\rangle \{\ell_3\ell_2\hat{\varepsilon}_3^{-1},\hat{\varepsilon}_3^{-1}\} W(\varepsilon_2)W(-1)\\
& && \qquad W(\ell_3)W(-1)\mathbf{X}_2\mathbf{X}_1\\
&\longrightarrow && \langle b,c\rangle \langle \theta_2^{-1}a\ell_2^{-1},\ell_2 c\ell_2^{-1}\rangle \{\ell_3\ell_2\hat{\varepsilon}_3^{-1},\hat{\varepsilon}_3^{-1}\}\{\ell_2,\ell_3\} WW(-1)\mathbf{X}_2\mathbf{X}_1\\
&\longrightarrow && \langle a+b,c\rangle WW(-1)\mathbf{X}_2\mathbf{X}_1
\end{aligned}
$$

where $\varepsilon_2 = 1+(b)(c)$, $\varepsilon_3 = 1+(\varepsilon_2^{-1}a\ell_2^{-1})(\ell_2 c)$.

[[4,6]](ii)

$$
\begin{aligned}
&(4) && \mathbf{X}_1(a)\mathbf{X}_2(b)\mathbf{X}_2(c)\\
&\longrightarrow && \mathbf{X}_1(a)\mathbf{X}_2(b+c)\\
&\longrightarrow && \langle a,b+c\rangle WW(-1)\mathbf{X}_2\mathbf{X}_1
\end{aligned}
$$

$$
\begin{aligned}
&(6) \quad X_1(a)X_2(b)X_2(c) \\
&\longrightarrow \langle a,b\rangle W(\ell_5)W(-1)X_2(\ell_5 b)X_1(\varepsilon_5^{-1}a)X_2(c) \\
&\longrightarrow \langle a,b\rangle W(\ell_5)W(-1)X_2(\ell_5 b)\langle \varepsilon_5^{-1}a,c\rangle W(\ell_6)W(-1)X_2X_1 \\
&\longrightarrow \langle a,b\rangle W(\ell_5)W(-1)\langle \varepsilon_5^{-1}a,c\rangle X_2(\ell_5 b\theta_6)W(\ell_6)W(-1)X_2X_1 \\
&\longrightarrow \langle a,b\rangle W(\ell_5)W(-1)\langle \varepsilon_5^{-1}a,c\rangle W(\ell_6)X_1(-\ell_6\ell_5 b\theta_6\ell_6)W(-1)X_2X_1 \\
&\longrightarrow \langle a,b\rangle W(\ell_5)W(-1)\langle \varepsilon_5^{-1}a,c\rangle W(\ell_6)W(-1)X_2X_1 \\
&\longrightarrow \langle a,b\rangle W(\ell_5)W(-1)\langle \varepsilon_5^{-1}a,c\rangle W(\ell_6)W(-1)X_2X_1 \\
&\longrightarrow \langle a,b\rangle \langle \ell_5\varepsilon_5^{-1}a, c\ell_5^{-1}\rangle \{\ell_6\ell_5\hat{\varepsilon}_6^{-1}, \hat{\varepsilon}_6^{-1}\}W(\ell_5)W(-1)W(\ell_6)W(-1)X_2X_1 \\
&\longrightarrow \langle a,b\rangle \langle \ell_5\varepsilon_5^{-1}a, c\ell_5^{-1}\rangle \{\ell_6\ell_5\hat{\varepsilon}_6^{-1}, \hat{\varepsilon}_6^{-1}\}\{\ell_5,\ell_6\}WW(-1)X_2X_1 \\
&\longrightarrow \langle a,b+c\rangle WW(-1)X_2X_1
\end{aligned}
$$

where $\varepsilon_4 = 1 + (a)(b+c)$, $\varepsilon_5 = 1+(a)(b)$, $\varepsilon_6 = 1+(\varepsilon_5^{-1}a)(c)$

[[4,10a]]

$$
\begin{aligned}
&(4) \quad X_1(c)X_1(d)\langle a,b\rangle \\
&\longrightarrow X_1(c+d)\langle a,b\rangle \\
&\longrightarrow \langle a,b\rangle X_1
\end{aligned}
$$

$$
\begin{aligned}
&(10a) \quad X_1(c)X_1(d)\langle a,b\rangle \\
&\longrightarrow \quad X_1(c)\langle a,b\rangle X_1 \\
&\longrightarrow \quad \langle a,b\rangle X_1
\end{aligned}
$$

[[4,10b]]

$$
\begin{aligned}
&(4) \quad X_2(c)X_2(d)\langle a,b\rangle \\
&\longrightarrow \quad X_2(c+d)\langle a,b\rangle \\
&\longrightarrow \quad \langle a,b\rangle X_2
\end{aligned}
$$

$$
\begin{aligned}
&(10b) \quad X_2(c)X_2(d)\langle a,b\rangle \\
&\longrightarrow \quad X_2(c)\langle a,b\rangle X_2 \\
&\longrightarrow \quad \langle a,b\rangle X_2
\end{aligned}
$$

[[5a,8]]

$$
\begin{aligned}
&(5a) \quad X_1(a)W(u)W(v) \\
&\longrightarrow \quad W(u)X_2(-u^{-1}au^{-1})W(v) \\
&\longrightarrow \quad W(u)W(v)X_1 \\
&\longrightarrow \quad \{-uv,-v^{-1}\}WW(-1)X_1
\end{aligned}
$$

(8) $X_1(a)W(u)W(v)$

$\longrightarrow$ $X_1(a)\{-uv,-v^{-1}\}W(-v^{-1}u)W(-1)$

$\longrightarrow$ $\{-uv,-v^{-1}\}X_1WW(-1)$

$\longrightarrow$ $\{-uv,-v^{-1}\}WW(-1)X_1$

[[5a,9]] (5a) $X_1(a)W(u)W(-1)W(v)$

$\longrightarrow$ $W(u)X_2W(-1)W(v)$

$\longrightarrow$ $W(u)W(-1)W(v)X_2$

$\longrightarrow$ $\{u,v\}WX_2$

(9) $X_1(a)W(u)W(-1)W(v)$

$\longrightarrow$ $X_1(a)\{u,v\}W(vu)$

$\longrightarrow$ $\{u,v\}X_1W(vu)$

$\longrightarrow$ $\{u,v\}WX_2$

[[5a,11]] (5a) $X_1(c)W(u)\langle a,b\rangle$

$\longrightarrow$ $W(u)X_2(-u^{-1}cu^{-1})\langle a,b\rangle$

$\longrightarrow$ $W(u)\langle a,b\rangle X_2$

$\longrightarrow$ $\langle -u^{-1}au^{-1},-ubu\rangle\{\theta_7^{-1}\ell_7,\ell_7^{-1}\}\{-\varepsilon_7^{-1},-u\}$
$\{-u\varepsilon_7^{-1},-\ell_7\}WX_2$

(11) $X_1(c)W(u)\langle a,b\rangle$

$\longrightarrow$ $X_1(c)\langle -u^{-1}au^{-1},-ubu\rangle\{\theta_7^{-1}\ell_7,\ell_7^{-1}\}\{-\varepsilon_7^{-1},-u\}$
$\{-u\varepsilon_7^{-1},-\ell_7\}W$

$\longrightarrow$ $\langle -u^{-1}au^{-1},-ubu\rangle\{\theta_7^{-1}\ell_7,\ell_7^{-1}\}\{-\varepsilon_7^{-1},-u\}$
$\{-u\varepsilon_7^{-1},-\ell_7\}WX_2\varepsilon_7 = 1+(-u^{-1}au^{-1})(-ubu)$

where $\varepsilon_7 = 1 + (u^{-1}au^{-1})(ubu)$

[[5b,6]] (5b) $X_1(a)X_2(b)W(u)$

$\longrightarrow$ $X_1(a)W(u)X_1(-ubu)$

$\longrightarrow$ $WX_2X_1$

$$
\begin{array}{ll}
(6) & \mathbf{X}_1(a)\mathbf{X}_2(b)W(u) \\
\longrightarrow & \langle a,b\rangle W(\ell_5)W(-1)\mathbf{X}_2(\ell_5 b)\mathbf{X}_1(\varepsilon_5^{-1}a)W(u) \\
\longrightarrow & \langle a,b\rangle W(\ell_5)W(-1)W(u)\mathbf{X}_1(-u(\ell_5 b)u)\mathbf{X}_2(-u^{-1}\varepsilon_5^{-1}au^{-1}) \\
\longrightarrow & \langle a,b\rangle W(\ell_5)W(-1)W(u)\langle -u\ell_5 bu, -u^{-1}\varepsilon_5^{-1}au^{-1}\rangle \\
 & \qquad W(\ell_8)W(-1)\mathbf{X}_2\mathbf{X}_1 \\
\longrightarrow & \langle a,b\rangle \{\ell_5,u\}W(u\ell_5)\langle -u\ell_5 bu, -u^{-1}\varepsilon_5^{-1}au^{-1}\rangle W(\ell_8)W(-1) \\
 & \qquad \mathbf{X}_2\mathbf{X}_1 \\
\longrightarrow & \langle a,b\rangle \{\ell_5,u\}\langle bu(u\ell_5)^{-1}, u\ell_5 u^{-1}\varepsilon_5^{-1}a\ell_5\rangle \\
 & \qquad \{\theta_9^{-1}\ell_9,\ell_9^{-1}\}\{-\varepsilon_9^{-1},-u\ell_5\}\{-u\ell_5\varepsilon_9^{-1},-\ell_9\} \\
 & \qquad W(\ell_8 u\ell_5\varepsilon_9^{-1})W(\ell_8)W(-1)\mathbf{X}_2\mathbf{X}_1 \\
\longrightarrow & \langle a,b\rangle \{\ell_5,u\}\langle bu(u\ell_5)^{-1}, u\ell_5 u^{-1}\varepsilon_5^{-1}a\ell_5\rangle \\
 & \qquad \{\theta_9^{-1}\ell_9,\ell_9^{-1}\}\{-\varepsilon_9^{-1},-u\ell_5\}\{-u\ell_5\varepsilon_9^{-1},-\ell_9\} \\
 & \qquad \{-\ell_8 u\ell_5\varepsilon_9^{-1}\ell_8,-\ell_8^{-1}\}W(-u\ell_5\varepsilon_9^{-1}) \\
 & \qquad W(-1)W(-1)\mathbf{X}_2\mathbf{X}_1 \\
\longrightarrow & \langle a,b\rangle \{\ell_5,u\}\langle bu(u\ell_5)^{-1}, u\ell_5 u^{-1}\varepsilon_5^{-1}a\ell_5\rangle \\
 & \qquad \{\theta_9^{-1}\ell_9,\ell_9^{-1}\}\{-\varepsilon_9^{-1},-u\ell_5\}\{-u\ell_5\varepsilon_9^{-1},-\ell_9\} \\
 & \qquad \{-\ell_8 u\ell_5\varepsilon_9^{-1}\ell_8,-\ell_8^{-1}\}\{-u\ell_5\varepsilon_9^{-1},-1\} \\
 & \qquad W\mathbf{X}_2\mathbf{X}_1 \\
\longrightarrow & W(u)\mathbf{X}_2\mathbf{X}_1
\end{array}
$$

where $\varepsilon_8 = 1+(-u\ell_5 bu)(-u^{-1}\varepsilon_5^{-1}au^{-1})$, $\varepsilon_9 = 1+(bu(u\ell_5)^{-1})(u\ell_5 u^{-1}\varepsilon_5^{-1}a\ell_5)$

[[5b, 8]]

$$
\begin{array}{ll}
(5b) & \mathbf{X}_2(a)W(u)W(v) \\
\longrightarrow & W(u)\mathbf{X}_1 W(v) \\
\longrightarrow & W(u)W(v)\mathbf{X}_2 \\
\longrightarrow & \{-uv,-v^{-1}\}WW(-1)\mathbf{X}_2
\end{array}
$$

$$
\begin{array}{ll}
(8) & \mathbf{X}_2(a)W(u)W(v) \\
\longrightarrow & \mathbf{X}_2(a)\{-uv,-v^{-1}\}W(-v^{-1}u)W(-1) \\
\longrightarrow & \{-uv,-v^{-1}\}WW\mathbf{X}_2
\end{array}
$$

[[5b, 9]]

$$
\begin{array}{ll}
(5b) & \mathbf{X}_2(a)W(u)W(-1)W(v) \\
\longrightarrow & W(u)\mathbf{X}_1(-uau)W(-1)W(v) \\
\longrightarrow & W(u)W(-1)W(v)\mathbf{X}_1 \\
\longrightarrow & \{u,v\}W\mathbf{X}_1
\end{array}
$$

$$
\begin{aligned}
&(9) && \mathbf{X}_2(a)W(u)W(-1)W(v) \\
&\longrightarrow && \mathbf{X}_2(a)\{u,v\}W(vu) \\
&\longrightarrow && \{u,v\}W\mathbf{X}_1
\end{aligned}
$$

[[5b, 11]]

$$
\begin{aligned}
&(5b) && \mathbf{X}_2(c)W(u)\langle a,b\rangle \\
&\longrightarrow && W(u)\mathbf{X}_1\langle a,b\rangle \\
&\longrightarrow && W(u)\langle a,b\rangle\mathbf{X}_1 \\
&\longrightarrow && \langle -u^{-1}au^{-1},-ubu\rangle\{\theta_{10}{}^{-1}\varepsilon_{10},\varepsilon_{10}{}^{-1}\}\{-\epsilon_{10}{}^{-1},-u\}\{-u\epsilon_{10}{}^{-1},-\varepsilon_{10}\}W\mathbf{X}_1
\end{aligned}
$$

$$
\begin{aligned}
&(11) && \mathbf{X}_2(c)W(u)\langle a,b\rangle \\
&\longrightarrow && \mathbf{X}_2\langle -u^{-1}au^{-1},-ubu\rangle\{\theta_{10}{}^{-1}\varepsilon_{10},\varepsilon_{10}{}^{-1}\}\{-\epsilon_{10}{}^{-1},-u\}\{-u\epsilon_{10}{}^{-1},-\varepsilon_{10}\}W \\
&\longrightarrow && \langle -u^{-1}au^{-1},-ubu\rangle\{\theta_{10}{}^{-1}\varepsilon_{10},\varepsilon_{10}{}^{-1}\}\{-\epsilon_{10}{}^{-1},-u\}\{-u\epsilon_{10}{}^{-1},-\varepsilon_{10}\}W\mathbf{X}_1
\end{aligned}
$$

where $\epsilon_{10} = 1+(u^{-1}au^{-1})(ubu)$

[[6, 10b]]

$$
\begin{aligned}
&(6) && \mathbf{X}_1(c)\mathbf{X}_2(d)\langle a,b\rangle \\
&\longrightarrow && \langle c,d\rangle W(\varepsilon_{11})W(-1)\mathbf{X}_2(\varepsilon_{11}d)\mathbf{X}_1\langle a,b\rangle \\
&\longrightarrow && \langle c,d\rangle W(\varepsilon_{11})W(-1)\mathbf{X}_2(\varepsilon_{11}d)\langle a,b\rangle\mathbf{X}_1 \\
&\longrightarrow && \langle c,d\rangle W(\varepsilon_{11})W(-1)\langle a,b\rangle\mathbf{X}_2\mathbf{X}_1 \\
&\longrightarrow && \langle c,d\rangle\langle\varepsilon_{11}a,b\varepsilon_{11}{}^{-1}\rangle\{\varepsilon_5\varepsilon_{11}\hat{\varepsilon}_5^{-1},\hat{\varepsilon}_5^{-1}\}WW(-1)\mathbf{X}_2\mathbf{X}_1
\end{aligned}
$$

$$
\begin{aligned}
&(10b) && \mathbf{X}_1(c)\mathbf{X}_2(d)\langle a,b\rangle \\
&\longrightarrow && \mathbf{X}_1(c)\langle a,b\rangle\mathbf{X}_2(d\theta_5) \\
&\longrightarrow && \langle a,b\rangle\mathbf{X}_1(\theta_5{}^{-1}c)\mathbf{X}_2(d\theta_5) \\
&\longrightarrow && \langle a,b\rangle\langle\theta_5{}^{-1}c,d\theta_5\rangle WW(-1)\mathbf{X}_2\mathbf{X}_1 \\
&\longrightarrow && \langle c,d\rangle\langle\varepsilon_{11}a,b\varepsilon_{11}{}^{-1}\rangle\{\varepsilon_{30}\varepsilon_{11}\hat{\varepsilon}_{30}^{-1},\hat{\varepsilon}_{30}^{-1}\}WW(-1)\mathbf{X}_2\mathbf{X}_1
\end{aligned}
$$

where $\epsilon_{11} = 1+(c)(d)$, $\epsilon_{12} = 1+(\theta_5{}^{-1}c)(d\theta_5)$

[[8, 8]]

$$
\begin{aligned}
&(8) && W(u)W(v)W(z) \\
&\longrightarrow && \{-uv,-v^{-1}\}W(-v^{-1}u)W(-1)W(z)
\end{aligned}
$$

$$\longrightarrow \quad \{-uv,-v^{-1}\}\{-v^{-1}u,z\}W$$

$$
\begin{aligned}
&(8) && W(u)W(v)W(z)\\
&\longrightarrow && W(u)\{-vz,-z^{-1}\}W(-z^{-1}v)W(-1)\\
&\longrightarrow && \langle -u^{-1}vz(z^{-1}+1)u^{-1},u(vz)^{-1}u\rangle\{\varepsilon_{14},\ell_{14}^{-1}\}\{-\varepsilon_{14}^{-1},-u\}\\
& && \quad \{-u\varepsilon_{14}^{-1},-\ell_{14}\}W(\ell_{13}u\varepsilon_{14}^{-1})W(-z^{-1}v)W(-1)\\
&\longrightarrow && \langle -u^{-1}vz(z^{-1}+1)u^{-1},u(vz)^{-1}u\rangle\{\varepsilon_{14},\ell_{14}^{-1}\}\{-\varepsilon_{14}^{-1},-u\}\\
& && \quad \{-u\varepsilon_{14}^{-1},-\ell_{14}\}\{\ell_{13}u\varepsilon_{14}^{-1}z^{-1}v,(z^{-1}v)^{-1}\}\\
& && \quad W((z^{-1}v)^{-1}\ell_{13}u\varepsilon_{14}^{-1})W(-1)W(-1)\\
&\longrightarrow && \langle -u^{-1}vz(z^{-1}+1)u^{-1},u(vz)^{-1}u\rangle\{\varepsilon_{14},\ell_{14}^{-1}\}\{-\varepsilon_{14}^{-1},-u\}\\
& && \quad \{-u\varepsilon_{14}^{-1},-\ell_{14}\}\{\ell_{13}u\varepsilon_{14}^{-1}z^{-1}v,(z^{-1}v)^{-1}\}\\
& && \quad \{(z^{-1}v)^{-1}\ell_{13}u\varepsilon_{14}^{-1},-1\}W\\
&\longrightarrow && \{-uv,-v^{-1}\}\{-v^{-1}u,z\}W
\end{aligned}
$$

where $\theta_{13} = [v,z^{-1}]$, $\varepsilon_{14} = 1+(-u^{-1}v(1+z)u^{-1})(u(vz)^{-1}u)$.

[[8,9]] (i)

$$
\begin{aligned}
&(8) && W(u)W(v)W(-1)W(z)\\
&\longrightarrow && \{-uv,-v^{-1}\}W(-v^{-1}u)W(-1)W(-1)W(z)\\
&\longrightarrow && \{-uv,-v^{-1}\}\{-v^{-1}u,-1\}W(v^{-1}u)W(z)\\
&\longrightarrow && \{-uv,-v^{-1}\}\{-v^{-1}u,-1\}\{-v^{-1}uz,-z^{-1}\}WW(-1)
\end{aligned}
$$

$$
\begin{aligned}
&(9) && W(u)W(v)W(-1)W(z)\\
&\longrightarrow && W(u)\{v,z\}W(zv)\\
&\longrightarrow && \langle -u^{-1}v(z-1)u^{-1},-uv^{-1}u\rangle\{\theta_{16}^{-1}\ell_{16},\ell_{16}^{-1}\}\{-\varepsilon_{16}^{-1},-u\}\\
& && \quad \{-u\varepsilon_{16}^{-1},-\ell_{16}\}W(\ell_{15}u\varepsilon_{16}^{-1})W(zv)\\
&\longrightarrow && \langle -u^{-1}v(z-1)u^{-1},-uv^{-1}u\rangle\{\theta_{16}^{-1}\ell_{16},\ell_{16}^{-1}\}\{-\varepsilon_{16}^{-1},-u\}\\
& && \quad \{-u\varepsilon_{16}^{-1},-\ell_{16}\}\{-\ell_{15}u\varepsilon_{16}^{-1}zv,-(zv)^{-1}\}WW(-1)\\
&\longrightarrow && \{-uv,-v^{-1}\}\{-v^{-1}u,-1\}\{-v^{-1}uz,-z^{-1}\}WW(-1)
\end{aligned}
$$

where $\theta_{15} = [v,z]$, $\varepsilon_{16} = 1+(u^{-1}v(z-1)u^{-1})(uv^{-1}u)$.

[[8,9]] (ii)

$$
\begin{aligned}
&(9) && W(u)W(-1)W(v)W(z)\\
&\longrightarrow && \{u,v\}W(vu)W(z)\\
&\longrightarrow && \{u,v\}\{-vuz,-z^{-1}\}WW(-1)
\end{aligned}
$$

$$
\begin{array}{ll}
(8) & W(u)W(-1)W(v)W(z) \\
\longrightarrow & W(u)\{v,-v^{-1}\}W(v^{-1})W(-1)W(z) \\
\longrightarrow & W(u)\{v,-v^{-1}\}\{v^{-1},z\}W(zv^{-1}) \\
\longrightarrow & \langle u^{-1}(1+v)u^{-1},-uv^{-1}u\rangle\{\theta_{19}{}^{-1}\ell_{19},\ell_{19}{}^{-1}\} \\
 & \quad \{-\epsilon_{19}{}^{-1},-u\}\{-u\epsilon_{19}{}^{-1},-\ell_{19}\}W(u_2)\{v^{-1},z\}W(zv^{-1}) \\
\longrightarrow & \langle u^{-1}(1+v)u^{-1},-uv^{-1}u\rangle\{\theta_{19}{}^{-1}\ell_{19},\ell_{19}{}^{-1}\}\{-\epsilon_{19}{}^{-1},-u\} \\
 & \quad \{-u\epsilon_{19}{}^{-1},-\ell_{19}\}\langle u_2{}^{-1}v^{-1}(1-z)u_2{}^{-1},-u_2vu_2\rangle \\
 & \quad \{\theta_{19}{}^{-1}\ell_{19},\ell_{19}{}^{-1}\}\{-\epsilon_{19}{}^{-1},-u_2\}\{-u_2\epsilon_{19}{}^{-1},-\ell_{19}\} \\
 & \quad\quad W(\ell_{17}u_2\epsilon_{19}{}^{-1})W(zv^{-1}) \\
\longrightarrow & \langle u^{-1}(1+v)u^{-1},-uv^{-1}u\rangle\{\theta_{19}{}^{-1}\ell_{19},\ell_{19}{}^{-1}\} \\
 & \{-\epsilon_{19}{}^{-1},-u\}\{-u\epsilon_{19}{}^{-1},-\ell_{19}\}\langle -u_2{}^{-1}v^{-1}(z-1)u_2{}^{-1},-u_2vu_2\rangle \\
 & \{\theta_{19}{}^{-1}\ell_{19},\ell_{19}{}^{-1}\}\{-\epsilon_{19}{}^{-1},-u_2\}\{-u_2\epsilon_{19}{}^{-1},-\ell_{19}\} \\
 & \{-\ell_{17}u_2\epsilon_{19}{}^{-1}zv^{-1},-vz^{-1}\}WW(-1) \\
\longrightarrow & \{u,v\}\{-vuz,-z^{-1}\}WW(-1)
\end{array}
$$

where $\theta_{17} = \{v^{-1},z\}$, $\epsilon_{19} = 1+(-u^{-1}(1+v)u^{-1})(uv^{-1}u)$, $\epsilon_{19} = 1+(u_2{}^{-1}v^{-1}(z-1)u_2{}^{-1})(u_2vu_2)$, and $u_2 = \ell_{38}u\epsilon_{19}^{-1}$

$$
\begin{array}{lll}
[[8,9]] & (9) & W(u)W(-1)W(v)W(z) \\
(iii) & \longrightarrow & \{u,v\}W(vu)W(z) \\
 & \longrightarrow & \{u,v\}\{-vuz,-z^{-1}\}W(-z^{-1}vu)W(-1)
\end{array}
$$

$$
\begin{array}{ll}
(8) & W(u)W(-1)W(v)W(z) \\
\longrightarrow & W(u)W(-1)\{-vz,-z^{-1}\}W(-z^{-1}v)W(-1) \\
\longrightarrow & W(u)W(-1)\langle v(1+z),-z^{-1}v\rangle W(-z^{-1}v)W(-1) \\
\longrightarrow & \langle uv(1+z),-z^{-1}vu^{-1}\rangle\{\ell_{13}u\hat{\epsilon}_{13}^{-1},\hat{\epsilon}_{13}^{-1}\}W(u)W(-1)W(-z^{-1}v)W(-1) \\
\longrightarrow & \langle uv(1+z),-z^{-1}vu^{-1}\rangle\{\ell_{13}u\hat{\epsilon}_{13}^{-1},\hat{\epsilon}_{13}^{-1}\}\{u,-z^{-1}v\}WW(-1) \\
\longrightarrow & \{u,v\}\{-vuz,-z^{-1}\}WW(-1)
\end{array}
$$

$$
\begin{array}{lll}
[[8,9]] & (8) & W(u)W(v)W(-1)W(z) \\
(iv) & \longrightarrow & W(u)W(v)\{z,-z^{-1}\}W(z^{-1})W(-1) \\
 & \longrightarrow & \{-uv,-v^{-1}\}W(-v^{-1}u)W(-1)\{z,-z^{-1}\}W(z^{-1})W(-1) \\
 & \longrightarrow & \{-uv,-v^{-1}\}\langle(v^{-1}u)(1+z),-z^{-1}u^{-1}v\rangle\{-z^{-1}v^{-1}uz,-z\} \\
 & & \quad\quad W(-v^{-1}u)W(-1)W(z^{-1})W(-1) \\
 & \longrightarrow & \{-uv,-v^{-1}\}\langle(v^{-1}u)(1+z),-z^{-1}u^{-1}v\rangle\{-z^{-1}v^{-1}uz,-z\}
\end{array}
$$

$$\{-v^{-1}u, z^{-1}\}WW(-1)$$

where we note $1 + (-1-z)(z^{-1}) = 1 - z^{-1} - 1 = -z^{-1}$

and $1 + (z^{-1})(-1-z) = 1 - z^{-1} - 1 = -z^{-1}$

$$
\begin{array}{ll}
(9) & W(u)W(v)W(-1)W(z) \\
\longrightarrow & W(u)\{v,z\}W(zv) \\
\longrightarrow & W(u)\langle v(z-1), v^{-1}\rangle W(zv) \\
\longrightarrow & \langle -u^{-1}v(z-1)u^{-1}, -uv^{-1}u\rangle\{\theta_{16}^{-1}\varepsilon_{16}, \varepsilon_{16}^{-1}\}\{-\epsilon_{16}^{-1}, -u\} \\
& \quad \{-u\epsilon_{16}^{-1}, -\varepsilon_{16}\}W(\varepsilon_{15}u\epsilon_{16}^{-1})W(zv) \\
\longrightarrow & \langle -u^{-1}v(z-1)u^{-1}, -uv^{-1}u\rangle\{\theta_{16}^{-1}\varepsilon_{16}, \varepsilon_{16}^{-1}\}\{-\epsilon_{16}^{-1}, -u\} \\
& \quad \{-u\epsilon_{16}^{-1}, -\varepsilon_{16}\}\{-\varepsilon_{15}u\epsilon_{16}^{-1}zv, -(zv)^{-1}\}WW(-1) \\
\longrightarrow & \{-uv, -v^{-1}\}\langle (v^{-1}u)(1+z), -z^{-1}u^{-1}v\rangle\{-z^{-1}v^{-1}uz, -z\} \\
& \quad \{-v^{-1}u, z^{-1}\}WW(-1)
\end{array}
$$

[[8, 11]]

$$
\begin{array}{ll}
(8) & W(u)W(v)\langle a, b\rangle \\
\longrightarrow & \{-uv, -v^{-1}\}W(-v^{-1}u)W(-1)\langle a, b\rangle \\
\longrightarrow & \{-uv, -v^{-1}\}\langle -v^{-1}ua, -bu^{-1}v\rangle\{-\varepsilon_5 v^{-1}u\hat{\epsilon}_5^{-1}, \hat{\epsilon}_5^{-1}\}WW(-1)
\end{array}
$$

$$
\begin{array}{ll}
(11) & W(u)W(v)\langle a, b\rangle \\
\longrightarrow & W(u)\langle -v^{-1}av^{-1}, -vbv\rangle\{\theta_{20}^{-1}\varepsilon_{20}, \varepsilon_{20}^{-1}\}\{-\epsilon_{20}^{-1}, -v\} \\
& \quad \{-v\epsilon_{20}^{-1}, -\varepsilon_{20}\}W(\varepsilon_5 v\epsilon_{20}^{-1}) \\
\longrightarrow & \langle a_2, b_2\rangle\{\theta_{24}^{-1}\varepsilon_{24}, \varepsilon_{24}^{-1}\}\{-\epsilon_{24}^{-1}, -u\}\{-u\epsilon_{24}^{-1}, -\varepsilon_{24}\} \\
& W(u_3)\{\theta_{20}^{-1}\varepsilon_{20}, \varepsilon_{20}^{-1}\}\{-\epsilon_{20}^{-1}, -v\}\{-v\epsilon_{20}^{-1}, -\varepsilon_{20}\}W(\varepsilon_5 v\epsilon_{20}^{-1}) \\
\longrightarrow & \langle a_2, b_2\rangle\{\theta_{24}^{-1}\varepsilon_{24}, \varepsilon_{24}^{-1}\}\{-\epsilon_{24}^{-1}, -u\}\{-u\epsilon_{24}^{-1}, -\varepsilon_{24}\} \\
& \langle -u_3^{-1}\theta_{20}^{-1}(1-\varepsilon_{20})u_3^{-1}, -u_3\varepsilon_{20}^{-1}\theta_{20}u_3\rangle \\
& \{\theta_{28}^{-1}\varepsilon_{28}, \varepsilon_{28}^{-1}\}\{-\epsilon_{28}^{-1}, -u_3\}\{-u_3\epsilon_{28}^{-1}, -\varepsilon_{28}\} \\
& W(u_4)\{-\epsilon_{20}^{-1}, -v\}\{-v\epsilon_{20}^{-1}, -\varepsilon_{20}\}W(\varepsilon_5 v\epsilon_{20}^{-1}) \\
\longrightarrow & \langle a_2, b_2\rangle\{\theta_{24}^{-1}\varepsilon_{24}, \varepsilon_{24}^{-1}\}\{-\epsilon_{24}^{-1}, -u\}\{-u\epsilon_{24}^{-1}, -\varepsilon_{24}\} \\
& \langle -u_3^{-1}\theta_{20}^{-1}(1-\varepsilon_{20})u_3^{-1}, -u_3\varepsilon_{20}^{-1}\theta_{20}u_3\rangle\{\theta_{28}^{-1}\varepsilon_{28}, \varepsilon_{28}^{-1}\} \\
& \{-\epsilon_{28}^{-1}, -u_3\}\{-u_3\epsilon_{28}^{-1}, -\varepsilon_{28}\} \\
& \langle u_4^{-1}\epsilon_{20}^{-1}(-v-1)u_4^{-1}, u_4\epsilon_{20}u_4\rangle\{\theta_{32}^{-1}\varepsilon_{32}, \varepsilon_{32}^{-1}\} \\
& \{-\epsilon_{32}^{-1}, -u_4\}\{-u_4\epsilon_{32}^{-1}, -\varepsilon_{32}\}W(u_6)\{-v\epsilon_{20}^{-1}, -\varepsilon_{20}\} \\
& W(\varepsilon_5 v\epsilon_{20}^{-1}) \\
\longrightarrow & \langle a_2, b_2\rangle\{\theta_{24}^{-1}\varepsilon_{24}, \varepsilon_{24}^{-1}\}\{-\epsilon_{24}^{-1}, -u\}\{-u\epsilon_{24}^{-1}, -\varepsilon_{24}\}
\end{array}
$$

$$\langle -u_3^{-1}\theta_{20}^{-1}(1-\ell_{20})u_3^{-1}, -u_3\ell_{20}^{-1}\theta_{20}u_3\rangle$$
$$\{\theta_{28}^{-1}\ell_{28}, \ell_{28}^{-1}\}\{-\varepsilon_{28}^{-1}, -u_3\}\{-u_3\varepsilon_{28}^{-1}, -\ell_{28}\}$$
$$\langle u_4^{-1}\varepsilon_{20}^{-1}(-v-1)u_4^{-1}, u_4\varepsilon_{20}u_4\rangle\{\theta_{32}^{-1}\ell_{32}, \ell_{32}^{-1}\}$$
$$\{-\varepsilon_{32}^{-1}, -u_4\}\{-u_4\varepsilon_{32}^{-1}, -\ell_{32}\}$$
$$\langle u_6^{-1}v\varepsilon_{20}^{-1}(-\ell_{20}-1)u_6^{-1}, u_6\varepsilon_{20}vu_6\rangle$$
$$\{\theta_{36}^{-1}\ell_{36}, \ell_{36}^{-1}\}\{-\varepsilon_{36}^{-1}, -u_6\}\{-u_6\varepsilon_{36}^{-1}, -\ell_{36}\}$$
$$W(u_5)W(\ell_5v\varepsilon_{20}^{-1})$$
$$\longrightarrow \quad \langle a_2, b_2\rangle\{\theta_{24}^{-1}\ell_{24}, \ell_{24}^{-1}\}\{-\varepsilon_{24}^{-1}, -u\}\{-u\varepsilon_{24}^{-1}, -\ell_{24}\}$$
$$\langle -u_3^{-1}\theta_{20}^{-1}(1-\ell_{20})u_3^{-1}, -u_3\ell_{20}^{-1}\theta_{20}u_3\rangle$$
$$\{\theta_{28}^{-1}\ell_{28}, \ell_{28}^{-1}\}\{-\varepsilon_{28}^{-1}, -u_3\}\{-u_3\varepsilon_{28}^{-1}, -\ell_{28}\}$$
$$\langle u_4^{-1}\varepsilon_{20}^{-1}(-v-1)u_4^{-1}, u_4\varepsilon_{20}u_4\rangle\{\theta_{32}^{-1}\ell_{32}, \ell_{32}^{-1}\}$$
$$\{-\varepsilon_{32}^{-1}, -u_4\}\{-u_4\varepsilon_{32}^{-1}, -\ell_{32}\}$$
$$\langle u_6^{-1}v\varepsilon_{20}^{-1}(-\ell_{20}-1)u_6^{-1}, u_6\varepsilon_{20}vu_6\rangle$$
$$\{\theta_{36}^{-1}\ell_{36}, \ell_{36}^{-1}\}\{-\varepsilon_{36}^{-1}, -u_6\}\{-u_6\varepsilon_{36}^{-1}, -\ell_{36}\}$$
$$\{-u_5\ell_5v\varepsilon_{20}^{-1}, -(\ell_5v\varepsilon_{20}^{-1})^{-1}\}WW(-1)$$
$$\longrightarrow \quad \{-uv, -v^{-1}\}\langle -v^{-1}ua, -bu^{-1}v\rangle\{-\ell_5v^{-1}u\hat{\varepsilon}_5^{-1}, \hat{\varepsilon}_5^{-1}\}WW(-1)$$

where

$\varepsilon_{20} = 1+(-v^{-1}av^{-1})(-vbv)$, $\theta_{21} = \theta_{20}^{-1}\hat{\varepsilon}_{20}^{-1}\varepsilon_{20}$, $\theta_{22} = [\varepsilon_{20}^{-1}, v]$

$\theta_{23} = [v\varepsilon_{20}^{-1}, \ell_{20}]$, $\varepsilon_{24} = 1+(a_2)(b_2)$, $\theta_{25} = \theta_{24}^{-1}\hat{\varepsilon}_{24}^{-1}\varepsilon_{24}$, $\theta_{26} = [\varepsilon_{24}^{-1}, u]$

$\theta_{27} = [u\varepsilon_{24}^{-1}, \ell_{24}]$, $\varepsilon_{28} = 1+(u_3^{-1}\theta_{20}^{-1}(1-\ell_{20})u_3^{-1})(u_3\ell_{20}^{-1}\theta_{20}u_3)$

$\theta_{29} = \theta_{28}^{-1}\hat{\varepsilon}_{28}^{-1}\varepsilon_{28}$, $\theta_{30} = [\varepsilon_{28}^{-1}, u_3]$, $\theta_{31} = [u_3\varepsilon_{28}^{-1}, \ell_{28}]$

$\varepsilon_{32} = 1+(u_4^{-1}\varepsilon_{20}^{-1}(-v-1)u_4^{-1})(u_4\varepsilon_{20}u_4)$, $\theta_{33} = \theta_{32}^{-1}\hat{\varepsilon}_{32}^{-1}\varepsilon_{32}$, $\theta_{34} = [\varepsilon_{32}^{-1}, u_4]$

$\theta_{35} = [u_4\varepsilon_{32}^{-1}, \ell_{32}]$, $\varepsilon_{36} = 1+(u_6^{-1}v\varepsilon_{20}^{-1}(-\ell_{20}-1)u_6^{-1})(u_6\varepsilon_{20}u_6)$

$\theta_{37} = \theta_{36}^{-1}\hat{\varepsilon}_{36}^{-1}\varepsilon_{36}$, $\theta_{38} = [\varepsilon_{36}^{-1}, u_6]$, $\theta_{39} = [u_6\varepsilon_{36}^{-1}, \ell_{36}]$

$\theta_{40} = [u_5\ell_{11}v\varepsilon_{20}^{-1}, (\ell_{11}v\varepsilon_{20}^{-1})^{-1}]$,

$u_3 = \ell_{20}u\varepsilon_{24}^{-1}$, $u_4 = \ell_{21}u_3\varepsilon_{28}^{-1}$, $u_5 = \ell_{23}u_6\varepsilon_{36}^{-1}$, $u_6 = \ell_{22}u_4\varepsilon_{32}^{-1}$

$a_2 = u^{-1}v^{-1}av^{-1}u^{-1}$, $b_2 = uvbvu$

[[9,9]] (i)

$$(9) \quad W(u)W(-1)W(v)W(-1)W(z)$$
$$\longrightarrow \quad \{u, v\}W(vu)W(-1)W(z)$$
$$\longrightarrow \quad \{u, v\}\{vu, z\}W$$

$$(9) \quad W(u)W(-1)W(v)W(-1)W(z)$$
$$\longrightarrow \quad W(u)W(-1)\{v, z\}W(zv)$$

$$
\begin{array}{ll}
\longrightarrow & W(u)W(-1)\langle v(z-1), v^{-1}\rangle W(zv) \\
\longrightarrow & \langle uv(z-1), v^{-1}u^{-1}\rangle\{\ell_{15}u\hat{\varepsilon}_{15}^{-1}, \hat{\varepsilon}_{15}^{-1}\}W(u)W(-1)W(zv) \\
\longrightarrow & \langle uv(z-1), v^{-1}u^{-1}\rangle\{\varepsilon_{15}u\hat{\varepsilon}_{15}^{-1}, \hat{\varepsilon}_{15}^{-1}\}\{u, zv\}W \\
\longrightarrow & \{u, v\}\{vu, z\}W
\end{array}
$$

$$
\begin{array}{lll}
[[9,9]] & (9) & W(u)W(-1)W(-1)W(v) \\
(ii) & \longrightarrow & \{u,-1\}W(-u)W(v) \\
 & \longrightarrow & \{u,-1\}\{uv,-v^{-1}\}WW(-1)
\end{array}
$$

$$
\begin{array}{ll}
(9) & W(u)W(-1)W(-1)W(v) \\
\longrightarrow & W(u)\{-1, v\}W(-v) \\
\longrightarrow & W(u)\langle 1-v, -1\rangle W(-v) \\
\longrightarrow & \langle u^{-1}(v-1)u^{-1}, u^2\rangle\{\theta_{41}{}^{-1}uvu^{-1}, uv^{-1}u^{-1}\}\{-u^{-1}v^{-1}u, -u\} \\
 & \{-v^{-1}u, -uvu^{-1}\}W(u)W(-v) \\
\longrightarrow & \langle u^{-1}(v-1)u^{-1}, u^2\rangle\{\theta_{41}{}^{-1}uvu^{-1}, uv^{-1}u^{-1}\}\{-u^{-1}v^{-1}u, -u\} \\
 & \{-v^{-1}u, -uvu^{-1}\}\{uv, v^{-1}\}WW(-1) \\
\longrightarrow & \{u,-1\}\{uv,-v^{-1}\}WW(-1)
\end{array}
$$

$$
\begin{aligned}
1 &+ (u^{-1}(v-1)u^{-1})(u^2) \\
&= 1 + (u^{-1}vu^{-1} - u^{-2})u^2 \\
&= 1 + u^{-1}vu - 1 \\
&= u^{-1}vu
\end{aligned}
$$

$$
\begin{aligned}
1 &+ (u^2)(u^{-1}(v-1)u^{-1}) \\
&= 1 + u^2(u^{-1}vu^{-1} - u^{-2}) \\
&= 1 + uvu^{-1} - 1 \\
&= uvu^{-1}
\end{aligned}
$$

$$\theta_{41} = (u^{-1}vu)(uvu^{-1})^{-1} = u^{-1}vu^2vu^{-1}$$

$$
\begin{array}{lll}
[[9,11]] & (9) & W(u)W(-1)W(v)\langle a, b\rangle \\
 & \longrightarrow & \{u, v\}W(vu)\langle a, b\rangle \\
 & \longrightarrow & \{u, v\}\langle -(vu)^{-1}a(vu)^{-1}, -vubvu\rangle\{\theta_{42}{}^{-1}\ell_{42}, \ell_{42}{}^{-1}\} \\
 & & \{-\varepsilon_{42}{}^{-1}, -vu\}\{-vu\varepsilon_{42}{}^{-1}, -\ell_{42}\}W(\ell_5 vu\varepsilon_{42}{}^{-1})
\end{array}
$$

$$
\begin{aligned}
(11)\quad & W(u)W(-1)W(v)\langle a,b\rangle \\
\longrightarrow\quad & W(u)W(-1)\langle -v^{-1}av^{-1}, -vbv\rangle\{\theta_{43}{}^{-1}\ell_{43}, \ell_{43}{}^{-1}\}\{-\epsilon_{43}{}^{-1}, -v\} \\
& \{-v\epsilon_{43}{}^{-1}, -\ell_{43}\}W(\ell_5 v\epsilon_{43}{}^{-1}) \\
\longrightarrow\quad & \langle -uv^{-1}av^{-1}, -vbvu^{-1}\rangle\{\ell_{43}u\hat{\epsilon}^{-1}_{43}, \hat{\epsilon}^{-1}_{43}\}W(u)W(-1) \\
& \langle \theta_{43}{}^{-1}(1-\ell_{43}), \ell_{43}{}^{-1}\theta_{43}\rangle\{-\epsilon_{43}{}^{-1}, -v\}\{-v\epsilon_{43}{}^{-1}, -\ell_{43}\} \\
& W(\ell_5 v\epsilon_{43}{}^{-1}) \\
\longrightarrow\quad & \langle -uv^{-1}av^{-1}, -vbvu^{-1}\rangle\{\ell_{43}u\hat{\epsilon}^{-1}_{43}, \hat{\epsilon}^{-1}_{43}\} \\
& \langle u\theta_{43}{}^{-1}(1-\ell_{43}), \ell_{43}{}^{-1}\theta_{43}u^{-1}\rangle\{\ell_{44}u\hat{\epsilon}^{-1}_{44}, \hat{\epsilon}^{-1}_{44}\}W(u)W(-1) \\
& \langle -\epsilon_{43}{}^{-1}(-v-1), -\epsilon_{43}\rangle\{-v\epsilon_{43}{}^{-1}, -\ell_{43}\}W(\ell_5 v\epsilon_{43}{}^{-1}) \\
\longrightarrow\quad & \langle -uv^{-1}av^{-1}, -vbvu^{-1}\rangle\{\ell_{43}u\hat{\epsilon}^{-1}_{43}, \hat{\epsilon}^{-1}_{43}\} \\
& \langle u\theta_{43}{}^{-1}(1-\ell_{43}), \ell_{43}{}^{-1}\theta_{43}u^{-1}\rangle\{\ell_{44}u\hat{\epsilon}^{-1}_{44}, \hat{\epsilon}^{-1}_{44}\} \\
& \langle -u\epsilon_{43}{}^{-1}(-v-1), -\epsilon_{43}u^{-1}\rangle\{\ell_{45}u\hat{\epsilon}^{-1}_{45}, \hat{\epsilon}^{-1}_{45}\} \\
& \langle uv\epsilon_{43}{}^{-1}(\ell_{43}+1), -\epsilon_{43}vu^{-1}\rangle\{\ell_{46}u\hat{\epsilon}^{-1}_{46}, \hat{\epsilon}^{-1}_{46}\} \\
& W(u)W(-1)W(\ell_5 v\epsilon_{43}{}^{-1}) \\
\longrightarrow\quad & \langle -uv^{-1}av^{-1}, -vbvu^{-1}\rangle\{\ell_{43}u\hat{\epsilon}^{-1}_{43}, \hat{\epsilon}^{-1}_{43}\} \\
& \langle u\theta_{43}{}^{-1}(1-\ell_{43}), \ell_{43}{}^{-1}\theta_{43}u^{-1}\rangle\{\ell_{44}u\hat{\epsilon}^{-1}_{44}, \hat{\epsilon}^{-1}_{44}\} \\
& \langle -u\epsilon_{43}{}^{-1}(-v-1), -\epsilon_{43}u^{-1}\rangle\{\ell_{45}u\hat{\epsilon}^{-1}_{45}, \hat{\epsilon}^{-1}_{45}\} \\
& \langle uv\epsilon_{43}{}^{-1}(\ell_{43}+1), -\epsilon_{43}vu^{-1}\rangle\{\ell_{46}u\hat{\epsilon}^{-1}_{46}, \hat{\epsilon}^{-1}_{46}\}\{u, \ell_5 v\epsilon_{43}{}^{-1}\}W \\
\longrightarrow\quad & \{u,v\}\langle -(vu)^{-1}a(vu)^{-1}, -vubvu\rangle\{\theta_{44}{}^{-1}\ell_{44}, \ell_{44}{}^{-1}\} \\
& \{-\epsilon_{44}{}^{-1}, -vu\}\{-vu\epsilon_{44}{}^{-1}, -\ell_{44}\}W
\end{aligned}
$$

where $\epsilon_{42} = 1+((vu)^{-1}a(vu)^{-1})(vubvu)$, $\epsilon_{43} = 1+(v^{-1}av^{-1})(vbv)$,
$\theta_{44} = [\theta_{43}{}^{-1}\ell_{43}, \ell_{43}{}^{-1}]$, $\theta_{45} = [\epsilon_{43}{}^{-1}, v]$, $\theta_{46} = [v\epsilon_{43}{}^{-1}, \ell_{43}]$

Thus, for all cases the Church-Rosser property is satisfied, and we have proved the uniqueness of minimal elements.

Note: We have repeatedly made use of the fact that the normal form for an element $w(u)w(-1)\langle a,b\rangle$ may be determined in the Steinberg group using either elementary relations, or identity P16 of Proposition 6.4. By uniqueness of the normal form at matrix level, we have an identity between the corresponding products of Dennis-Stein symbols in the Steinberg group. By including this as one of the defining relations in $D^*(R,I)$, we see that the process of using the elementary replacements followed by relation 18 is equivalent to applying the identity P16.

$$w(u)w(-1)\langle a,b\rangle = \langle uau, u^{-1}bu^{-1}\rangle\{u^{-1}\ell u, u\}\{\ell u\ell, \hat{\ell}^{-1}\}w(u)w(-1), \quad \ell = 1+ba$$

Relation 18 can thus be determined as follows.

With identity P16 in mind, we wish to impose in $\mathfrak{M}_\Delta$ the equivalence of

$$\begin{aligned}
& W(u)W(-1)\langle a,b\rangle \\
\longrightarrow\quad & W(u)\langle -a,-b\rangle\{\theta_5{}^{-1}\ell_5, \ell_5{}^{-1}\}\{\varepsilon_5{}^{-1}, -\ell_5\}W(-\theta_5{}^{-1}) \\
\longrightarrow\quad & \langle u^{-1}au^{-1}, ubu\rangle\{\theta_{47}{}^{-1}\ell_{47}, \ell_{47}{}^{-1}\}\{-\varepsilon_{47}{}^{-1}, -u\}\{-u\varepsilon_{47}{}^{-1}, -\ell_{47}\}W(u_7) \\
& \quad \{\theta_5{}^{-1}\ell_5, \ell_5{}^{-1}\}\{\varepsilon_5{}^{-1}, -\ell_5\}W(-\theta_5{}^{-1}) \\
\longrightarrow\quad & \langle u^{-1}au^{-1}, ubu\rangle\{\theta_{47}{}^{-1}\ell_{47}, \ell_{47}{}^{-1}\}\{-\varepsilon_{47}{}^{-1}, -u\}\{-u\varepsilon_{47}{}^{-1}, -\ell_{47}\} \\
& \quad \langle -u_7{}^{-1}\theta_5{}^{-1}\ell_5(\ell_5{}^{-1}-1)(u_7)^{-1}, -u_7(\theta_5{}^{-1}\ell_5)^{-1}u_7\rangle \\
& \quad \{\theta_{49}{}^{-1}\ell_{49}, \ell_{49}{}^{-1}\}\{-\varepsilon_{49}{}^{-1}, -u_7\}\{-u_7\varepsilon_{49}{}^{-1}, -\ell_{49}\} \\
& \quad \langle u_8{}^{-1}\varepsilon_5{}^{-1}(\ell_5+1)u_8{}^{-1}, -u_8\varepsilon_5u_8\rangle\{\theta_{51}{}^{-1}\ell_{51}, \ell_{51}{}^{-1}\}\{-\varepsilon_{51}{}^{-1}, -u_8\} \\
& \quad \{-u_8\varepsilon_{51}{}^{-1}, -\ell_{51}\}W(u_9)W(-\theta_5{}^{-1}) \\
\longrightarrow\quad & \langle u^{-1}au^{-1}, ubu\rangle\{\theta_{47}{}^{-1}\ell_{47}, \ell_{47}{}^{-1}\}\{-\varepsilon_{47}{}^{-1}, -u\}\{-u\varepsilon_{47}{}^{-1}, -\ell_{47}\} \\
& \quad \langle -u_7{}^{-1}\theta_5{}^{-1}\ell_5(\ell_5{}^{-1}-1)(u_7)^{-1}, -u_7(\theta_5{}^{-1}\ell_5)^{-1}u_7\rangle \\
& \quad \{\theta_{49}{}^{-1}\ell_{49}, \ell_{49}{}^{-1}\}\{-\varepsilon_{49}{}^{-1}, -u_7\}\{-u_7\varepsilon_{49}{}^{-1}, -\ell_{49}\} \\
& \quad \langle u_8{}^{-1}\varepsilon_5{}^{-1}(\ell_5+1)u_8{}^{-1}, -u_8\varepsilon_5u_8\rangle\{\theta_{51}{}^{-1}\ell_{51}, \ell_{51}{}^{-1}\}\{-\varepsilon_{51}{}^{-1}, -u_8\} \\
& \quad \{-u_8\varepsilon_{51}{}^{-1}, -\ell_{51}\}\{u_9\theta_5{}^{-1}, \theta_5\}W(u)W(-1)
\end{aligned}$$

(which is the normal form using only the elementary replacements to determine the appropriate product of Dennis-Stein symbols)

where $\varepsilon_{47} = 1+(u^{-1}au^{-1})(ubu)$, $\theta_{48} = [\theta_5{}^{-1}\ell_5, \ell_5{}^{-1}]$,
$\varepsilon_{49} = 1+(-u_7{}^{-1}\theta_5{}^{-1}(1-\ell_5)u_7{}^{-1})(-u_7(\theta_5{}^{-1}\ell_5)^{-1}u_7)$, $\theta_{50} = [\varepsilon_5{}^{-1}, -\ell_5]$,
$\varepsilon_{51} = 1+(u_8{}^{-1}\varepsilon_5{}^{-1}(\ell_5+1)u_8{}^{-1})(-u_8\varepsilon_5u_8)$
$u_7 = \ell_5u\varepsilon_{47}{}^{-1}$, $u_8 = \ell_{48}u_7\varepsilon_{49}{}^{-1}$, $u_9 = \ell_{50}u_8, \varepsilon_{51}{}^{-1}$

and the much simpler normal form proved earlier in $St'(2,R)$, which is

$$w(u)w(-1)\langle a,b\rangle$$
$$\longrightarrow \quad \langle uau,u^{-1}bu^{-1}\rangle\{u^{-1}\ell u,u\}\{\ell u\ell,\hat{\varepsilon}^{-1}\}w(u)w(-1).$$

Thus the relation which we have chosen to impose in $D^{*}(R,I)$ is relation 18,

$$\begin{aligned}
&\langle u^{-1}au^{-1},ubu\rangle\{\theta_{47}{}^{-1}\ell_{47},\ell_{47}{}^{-1}\}\{-\varepsilon_{47}{}^{-1},-u\}\{-u\varepsilon_{47}{}^{-1},-\ell_{47}\}\\
&\qquad\langle -u_7{}^{-1}\theta_5{}^{-1}\ell_5(\ell_5{}^{-1}-1)(u_7)^{-1},-u_7(\theta_5{}^{-1}\ell_5)^{-1}u_7\rangle\\
&\qquad\{\theta_{49}{}^{-1}\ell_{49},\ell_{49}{}^{-1}\}\{-\varepsilon_{49}{}^{-1},-u_7\}\{-u_7\varepsilon_{49}{}^{-1},-\ell_{49}\}\\
&\qquad\langle u_8{}^{-1}\varepsilon_5{}^{-1}(\ell_5+1)u_8{}^{-1},-u_8\varepsilon_5u_8\rangle\{\theta_{51}{}^{-1}\ell_{51},\ell_{51}{}^{-1}\}\{-\varepsilon_{51}{}^{-1},-u_8\}\\
&\qquad\{-u_8\varepsilon_{51}{}^{-1},-\ell_{51}\}\{u_9\theta_5{}^{-1},\theta_5\}\\
=\ &\langle uau,u^{-1}bu^{-1}\rangle\{u^{-1}\ell u,u\}\{\ell u\ell,\hat{\varepsilon}^{-1}\}
\end{aligned}$$

□

**Lemma 6.9** The following identities hold in every $\mathfrak{M}_\pi$ that contains each term of the expression:

(i) $X_i(a) * X_i(b) = X_i(a+b)$

(ii) $X_2(a) * W(u) = W(u) * X_1(-u^{-1}au^{-1})$

(iii) $X_1(c) * \langle a,b\rangle = \langle a,b\rangle * X_1(\theta^{-1}c)$

$\quad\;\; X_2(c) * \langle a,b\rangle = \langle a,b\rangle * X_2(c\theta)$

(iv) $\langle ay,b\rangle * \langle ba,y\rangle = \langle a,yb\rangle$

**Proof**: Follows directly from the elementary replacements.

□

**Lemma 6.10**: $\mathfrak{M}_\triangle$ is a normal subgroup of every $\mathfrak{M}_\pi$.

**Proof**: Every $\mathfrak{M}_\pi$ contains the group $\mathfrak{M}_\triangle$ where $\triangle$ is the order relation corresponding to the diagonal in $\{1,2\}\times\{1,2\}$, and $\mathfrak{M}_\triangle$ is a subgroup of every $\mathfrak{M}_\pi$, so we need only show normality. It is sufficient to see that:

(i) $X_i(a)DX_i(a)^{-1} \in \mathfrak{M}_\triangle,\ \forall\ i\in\{1,2\},\ a\in R,\ D\in D^{*}(R,I)$

(ii) $X_i(a)W(u)W(-1)X_i(a)^{-1} \in \mathfrak{M}_\Delta$, $\forall\ i \in \{1,2\}$, $a \in R$, $u \in 1+I$

and (iii) $X_i(a)X_j(b)X_i(a)^{-1} \in \mathfrak{M}_\Delta$, $\forall\ i,j \in \{1,2\}$, $a \in R$, $b \in I$,

and that $X(DWWX_2X_1)X$ corresponds to a minimal element with two W's.

Before we prove that these elements are in $\mathfrak{M}_\Delta$, we first note that for the symbol $\langle b,c\rangle \in D^+(R,I)$, we have $b \in I$ or $c \in I$, so that $bc, cb \in I$. Furthermore,

$$\begin{aligned} bc, cb \in I \quad &\Longrightarrow \quad 1 - (1+bc) \in I \\ &\Longrightarrow \quad [1-(1+bc)](1+bc)^{-1} \in I \\ &\Longrightarrow \quad (1+bc)^{-1} - 1 \in I \\ &\Longrightarrow \quad (1+bc)^{-1} + cb(1+bc)^{-1} - 1 \in I \\ &\Longrightarrow \quad (1+cb)(1+bc)^{-1} - 1 \in I \\ &\Longrightarrow \quad \theta^{-1} - 1 \in I \end{aligned}$$

and similarly, $bc, cb \in I \implies \theta - 1 \in I$.

Now, we see that (i) follows because

$$X_1(a)\langle b,c\rangle X_1(-a) = \langle b,c\rangle X_1((\theta^{-1}-1)a) \in \mathfrak{M}_\Delta$$

and $$X_2(a)\langle b,c\rangle X_2(-a) = \langle b,c\rangle X_1(a(\theta-1)) \in \mathfrak{M}_\Delta.$$

The second, (ii), follows by noting that

$X_1(a)W(u)W(-1)X_1(a)^{-1} = X_1(a-uau)W(u)W(-1) \in \mathfrak{M}_\Delta$, because $u \in 1+I$ implies $u=1+b$, some $b \in I$, so we have

$$\begin{aligned} a-uau &= a - (1+b)a(1+b) = a - a(1+b) - ba(1+b) \\ &= a - a - ab - ba - bab \in I, \end{aligned}$$

and $X_2(a)W(u)W(-1)X_2(a)^{-1} = W(u)W(-1)X_2(uau-a) \in \mathfrak{M}_\Delta$, because $a-uau \in I$ as above.

The third, (iii), is trivial when $i = j$, so we only need consider (for $b \in I$).

$$X_1(a)X_2(b)X_1(-a) = \langle a,b\rangle W(\ell)W(-1)X_2(\ell b)X_1(\epsilon^{-1}a-a) \in \mathfrak{M}_\triangle,$$
because $\epsilon^{-1}-1 \in I$

and
$$X_2(a)X_1(b)X_2(-a) = X_2(a)\langle b,-a\rangle W(\ell)W(-1)X_2(-\ell a)X_1(\epsilon^{-1}b)$$
$$= X_2(a)\langle b,-a\rangle X_2(-\hat{\epsilon}^{-1}\ell a\hat{\epsilon}^{-1})W(\ell)W(-1)X_1(\epsilon^{-1}b)$$
$$= X_2(a)X_2(-a\hat{\epsilon}^{-1}\theta^{-1})\langle b,-a\rangle W(\ell)W(-1)X_1(\epsilon^{-1}b) \in \mathfrak{M}_\triangle,$$
because $a-a\hat{\epsilon}^{-1}\theta^{-1} = a(1-\hat{\epsilon}^{-1}\ell\epsilon^{-1}) = a(1-\epsilon^{-1}) \in I$.

In addition, by straightforward application of the elementary replacements, it is easy to see that any element, $X(DWWXX)X$, corresponds to an element with two $W$'s. Thus $\mathfrak{M}_\triangle$ is normal in $\mathfrak{M}_\pi$

$\square$

This finishes completely the proof of the Theorem 6.1, and all necessary supporting lemmas

# APPENDIX

This appendix consists of a computer program used as an aid in the completion of the computations necessary to determine a statement of the 18 relations in Theorem 6.1. The program takes a series of letters, $X_1(a), X_2(a), W(u), \langle a, b\rangle, \{u, v\}$, and uses the elementary replacements defined in the proof of Theorem 6.1 (see Definition 6.5) to reduce the series of letters to the associated normal form. While the program uses a reasonably good decision process concerning the best route for reduction of the word to the normal form, in many cases, the process may be optimized greatly by an awareness of the interrelationships among the replacements. Thus, while this program is sufficient to independently generate the computations, the computations shown in this Dissertation are more refined. The program served as a valuable aid by allowing the author the opportunity to do a great deal of fine tuning by means of trial and error.

The program is written in the programming language of Pascal, and was developed and debugged using a combination of Apple Pascal, and Turbo Pascal, running on an Apple Macintosh Plus computer. The final version is written for Turbo Pascal. Due to constraints imposed by Turbo Pascal, the arrays are dimensioned too small for some of the longer computations, but these computations are easily broken up into smaller segments. By porting this program to other Pascal compilers, one may extend the array dimensions and do the computations each as a whole. On a non-Macintosh computer, one may choose to define a function that fulfills the need for superscripts and subscripts (done with a specially designed font in this version.) The structure of the program should allow it to be easily modified for use on other computations of a similar nature.

We conclude this appendix with the complete program listing.

```
program Compute,
 type
  argument = string[100];
  info = record
    l : string[1],
    s : array[1..2] of integer;
    a : array[1..2] of argument;
    e : integer
   end;
  anelist = array[1..45] of string[80],
  line = array[1..45] of info;

 var
  f : line,
  efirst, enext, last, laste : integer,
  done : boolean,
  elist : anelist;
  a:char,
  s1,filename : string,
  outfile                      : text;


 function dash (sign : integer) : string,
 begin
  if sign = -1 then
   dash := '-'
  else
   dash := ''
 end;

 function dashplus (sign : integer) : string;
 begin
  if sign = -1 then
   dashplus := '-'
  else
```

```
  dashplus := '+'
end;

function digit(n : integer):string;        {make into string}
begin
 if n < 10 then
  case n of
    1: digit:='1';
    2: digit:='2';
    3: digit:='3';
    4: digit:='4';
    5: digit:='5';
    6: digit:='6';
    7: digit:='7';
    8: digit:='8';
    9: digit:='9';
    end
end;

function str (n : integer) : string;        {make into string}
var
    strtemp : string;
begin
 if n < 10 then
    str := digit(n)
 else
    begin
      strtemp := concat(digit(n div 10), digit(n mod 10));
      if n mod 10 = 0 then
            strtemp := concat(digit(n div 10), digit(n mod
                                                        10),'0');
      str := strtemp;
    end;
end;
```

```
procedure dox1x2 (var f : line;
        var last, i,enext : integer);
  var
   j : integer;
   s1, s2 : string;
 begin
{copy i to last into i+3 to last+3}
  for j := last downto i do
   f[j + 3] := f[j];
  last := last + 3;
{create <a,b> at i}
  f[i].l := 'd';
  f[i].s[1] := f[i + 3].s[1];
  f[i].s[2] := f[i + 4].s[1];
  f[i].a[1] := f[i + 3].a[1];
  f[i].a[2] := f[i + 4].a[1];
{create w at i+1}
  f[i + 1].l := 'w';
  f[i + 1].s[1] := 1;
  f[i + 1].a[1] := concat('f', str(enext));
{create w at i+2}
  f[i + 2].l := 'w';
  f[i + 2].s[1] := -1;
  f[i + 2].a[1] := '1';
{create x2 at i+3}
  f[i + 3].l := '2';
  f[i + 3].s[1] := f[i].s[2];
  f[i + 3].a[1] := concat(f[i + 1].a[1], '[', f[i].a[2], ']');
{create x1 at i+4}
  f[i + 4].l := '1';
  f[i + 4].s[1] := f[i].s[1];
  f[i + 4].a[1] := concat('ε', str(enext), '^-1[', f[i].a[1], ']');
{create epsilon for <a,b> at i}
  f[i].e := enext;
  s1 := dash(f[i].s[1]);
```

```
  s2 := dash(f[i].s[2]);
  elist[enext] := concat('1+(', s1, f[i].a[1], ')(', s2, f[i].a[2],
                                                                  ')');

{next enext}
  enext := enext + 1; end;

 procedure getf (var f : line;
         var last : integer);              {input the orig line}
 begin
  f[1].l := 'w';
  f[1].s[1] := 1;
  f[1].a[1] := 'u';
  f[2].l := 'w';
  f[2].s[1] := 1;
  f[2].a[1] := 'v';
  f[2].s[2] := 1;
  f[2].a[2] := 'v';
  f[2].e := 1;
  f[3].l := '2';
  f[3].s[1] := 1;
  f[3].a[1] := 'c';
  f[4].l := 'w';
  f[4].s[1] := 1;
  f[4].a[1] := 'v';
  last := 2;
 end;

function takesign (var s:argument): integer;
 begin
   takesign := 1;
   if (length(s) >=1) and (copy(s,1,1) = '-')
   then begin
           takesign := -1;
           delete(s,1,1)
```

```
      end
end,

procedure find2 (let : string;
     {find a letter looking}
        f : line,
                      {right to left, and}
        last : integer;
          {skipping 'last'}
        var i : integer;
        var found : boolean);
begin
 found := false,
 i := last - 1,
 while (i >= 1) and (not found) do
  if f[i].l = let then
   found := true
  else
   i := i - 1,
end,

function pairxl (f : line;
        last : integer;
        var i : integer) : boolean;      {find a pair of xl's}
 var
  gotit : boolean;
begin
 pairxl := false;
 gotit := false;
 i := last;
 while (i > 1) and not gotit do
  begin
   if (f[i].l = 'l') and (f[i - 1].l = 'l') then
    begin
     pairxl := true,
```

```
      gotit := true
     end;
    i := i - 1;
   end;
  end;


function pairx2 (f : line;
        last : integer;
        var i : integer) : boolean;      {find a pair of x2's}
 var
  gotit : boolean;
begin
 pairx2 := false;
 gotit := false;
 i := last;
 while (i > 1) and not gotit do
  begin
   if (f[i].l = '2') and (f[i - 1].l = '2') then
    begin
     pairx2 := true;
     gotit := true
    end;
   i := i - 1;
  end;
end;

procedure dox1w (var f : line;
        var last,i,enext : integer);
var temp:info;
begin
 temp := f[i];
 f[i] := f[i+1];
 f[i+1].l := '2';
 f[i+1].s[1] := -1 * temp.s[1];
```

```
 f[i+1].a[1] := concat( '(',f[i].a[1],')^-1(',temp.a[1],')(',
                                                      f[i].a[1],')^-1'),
end;

procedure dox1d (var f : line;
        var last,i,enext : integer);
var temp : info;
begin
 temp := f[i];
 f[i] := f[i+1];
 f[i+1] := temp;
 f[i+1].a[1] := concat('θ',str(f[i].e),'^-1[',f[i+1].a[1],']');
end;

procedure dox1s (var f : line;
        var last,i,enext : integer);
var temp : info;
begin
 temp := f[i];
 f[i] := f[i+1];
 f[i+1] := temp;
 f[i+1].a[1] := concat('θ',str(f[i].e),'^-1[',f[i+1].a[1],']');
end;

procedure dox2w (var f : line;
        var last,i,enext : integer);
var temp:info;
begin
 temp := f[i];
 f[i] := f[i+1];
 f[i+1].l := '1';
 f[i+1].s[1] := -1 * temp.s[1];
 f[i+1].a[1] := concat( f[i].a[1],'(',temp.a[1],')',f[i].a[1]);
end;
```

```
procedure dox2d (var f : line,
        var last,i,enext : integer);
var temp : info,
begin
 temp := f[i];
 f[i] := f[i+1],
 f[i+1] := temp;
 f[i+1].a[1] := concat('[',f[i+1].a[1],']θ',str(f[i].e));
end,

procedure dox2s (var f : line,
        var last,i,enext : integer);
var temp : info,
begin
 temp := f[i],
 f[i] := f[i+1];
 f[i+1] := temp,
 f[i+1].a[1] := concat('[',f[i+1].a[1],']θ',str(f[i].e)),
end;

procedure doww(var f : line,
        var last,i,enext : integer);
var
  j: integer,
  s1,s2: string[5];

begin
{copy i to last into i+1 to last+1}
  for j := last downto i do
     f[j+1] := f[j],
  last := last + 1;
{create steinberg symbol at i}
  f[i].l := 's';
  f[i].s[1] := -1 * f[i + 1].s[1] * f[i + 2].s[1];
  f[i].s[2] := -1 * f[i + 2].s[1],
```

```
  f[i].a[1] := concat(f[i + 1].a[1],f[i + 2].a[1]);
  f[i].a[2] := concat('(',f[i + 2].a[1],')^-1');
{create w at i+1}
  f[i + 1].l := 'w';
  f[i + 1].s[1] := -1 * f[i + 2].s[1] * f[i + 1].s[1];
  f[i + 1].a[1] := concat('(',f[i + 2].a[1],')^-1',f[i + 1].a[1]);
{create w at i+2}
  f[i + 2].l := 'w';
  f[i + 2].s[1] := -1;
  f[i + 2].a[1] := '1';
{create epsilon for symbol at i}
  f[i].e := enext;
  s1 := dash(f[i].s[1]);
  s2 := dash(f[i].s[2]);
  elist[enext] := concat('[', s1, f[i].a[1], ',', s2, f[i].a[2],
                                                                    ']');
{next enext}
  enext := enext + 1;
end;

procedure dowww(var f : line;
          var last,i,enext : integer);
var
  j: integer;
  s1,s2: string[5];

begin
{copy i+2 to last into i+1 to last-1}
  for j := i+2 to last do
     f[j-1] := f[j];
  last := last - 1;
{create steinberg symbol at i}
  f[i].l := 's';
  f[i].s[1] := f[i].s[1];
  f[i].s[2] := f[i + 1].s[1];
```

```
    f[i].a[1] := f[i].a[1];
    f[i].a[2] := f[i + 1].a[1];
  {create epsilon for symbol at i}
    f[i].e := enext;
    s1 := dash(f[i].s[1]);
    s2 := dash(f[i].s[2]);
    elist[enext] := concat('[', s1, f[i].a[1], ',', s2, f[i].a[2],
                                                                   ']');
  {next enext}
    enext := enext + 1;
  {create w at i+1}
    f[i + 1].l := 'w';
    f[i + 1].s[1] := f[i].s[1] * f[i].s[2];
    f[i + 1].a[1] := concat(f[i].a[2],f[i].a[1]);
  end;

  procedure dowd(var f : line;
            var last,i,enext : integer);
  var
     a,b,u :     string;
     s1,s2,z,zprev : string[5];
     sa,sb,su,j : integer;

  begin
    a:=f[i+1].a[1]; sa:=f[i+1].s[1];
    b:=f[i+1].a[2]; sb:=f[i+1].s[2];
    u:=f[i].a[1]; su:=f[i].s[1];
    zprev := str(f[i+1].e);
  {copy i to last into i+3 to last+3}
    for j := last downto i do
       f[j+3] := f[j];
    last := last + 3;
  {create a D-S symbol at i}
    f[i].l := 'd';
    f[i].s[1] := -1 * sa;
```

```
  f[i].s[2] := -1 * sb;
  f[i].a[1] := concat('(',u,')^-1',a,'(',u,')^-1');
  f[i].a[2] := concat(u,b,u);
{create epsilon for <a,b> at i}
  f[i].e := enext;
  s1 := dash(f[i].s[1]);
  s2 := dash(f[i].s[2]);
  elist[enext] := concat('1+(', s1, f[i].a[1], ')(', s2, f[i].a[2],
                                                                  ')');
  z := str(enext);
{next enext}
  enext := enext + 1;
{create a Steinberg symbol at i+1}
  f[i+1].l := 's';
  f[i+1].s[1] := 1;
  f[i+1].s[2] := 1;
  f[i+1].a[1] := concat('θ',z,'^-1ε',z);
  f[i+1].a[2] := concat('ε',z,'^-1');
{create epsilon for symbol at i+1}
  f[i+1].e := enext;
  s1 := dash(f[i+1].s[1]);
  s2 := dash(f[i+1].s[2]);
  elist[enext] := concat('[', s1, f[i+1].a[1], ',', s2, f[i+1].a[2],
                                                                  ']');
{next enext}
  enext := enext + 1;
{create a Steinberg symbol at i+2}
  f[i+2].l := 's';
  f[i+2].s[1] := -1;
  f[i+2].s[2] := -1 * su;
  f[i+2].a[1] := concat('ε',z,'^-1');
  f[i+2].a[2] := u;
{create epsilon for symbol at i+2}
  f[i+2].e := enext;
  s1 := dash(f[i+2].s[1]);
```

```
  s2 := dash(f[i+2].s[2]);
  elist[enext] := concat('[', s1, f[i+2].a[1], ',', s2, f[i+2].a[2],
                                                                  ']');
{next enext}
  enext := enext + 1;
{create a Steinberg symbol at i+3}
  f[i+3].l := 's';
  f[i+3].s[1] := -1 * su;
  f[i+3].s[2] := -1;
  f[i+3].a[1] := concat(u,'ε',z,'-1');
  f[i+3].a[2] := concat('ℓ',z);
{create epsilon for symbol at i+3}
  f[i+3].e := enext;
  s1 := dash(f[i+3].s[1]);
  s2 := dash(f[i+3].s[2]);
  elist[enext] := concat('[', s1, f[i+3].a[1], ',', s2, f[i+3].a[2],
                                                                  ']');
{next enext}
  enext := enext + 1;
{create w at i+4}
  f[i + 4].l := 'w';
  f[i + 4].s[1] := su;
  f[i + 4].a[1] := concat('ℓ',zprev,u,'ε',z,'-1');
end;

procedure dows(var f : line;
         var last,i,enext : integer);

{just change the  Steinberg symbol at i+1 to a D-S}

var
   a,b :    string;
   s1,s2 : string[5];
   sa,sb,j : integer;
```

```
begin
{change to D-S symbol and then use dowd automatically next round}
  a:=f[i+1].a[1]; sa:=f[i+1].s[1];
  b:=f[i+1].a[2]; sb:=f[i+1].s[2];
{create a D-S symbol at i+1}
  f[i+1].l := 'd';
  f[i+1].s[1] := sa;
  f[i+1].s[2] := sa;
  f[i+1].a[1] := concat(a,'(',dash(sb),b,'-1)');
  f[i+1].a[2] := concat('(',a,')^-1');
{epsilon stays as is}
end;

procedure dohs(var f : line;
          var last,i,enext : integer);

{just change the  Steinberg symbol at i+1 to a D-S}

var
   a,b :     string;
   s1,s2 : string[5];
   sa,sb,j : integer;

begin
{change to D-S symbol and then use dowd automatically next round}
  a:=f[i+1].a[1]; sa:=f[i+1].s[1];
  b:=f[i+1].a[2]; sb:=f[i+1].s[2];
{create a D-S symbol at i+1}
  f[i+1].l := 'd';
  f[i+1].s[1] := sa;
  f[i+1].s[2] := sa;
  f[i+1].a[1] := concat(a,'(',dash(sb),b,'-1)');
  f[i+1].a[2] := concat('(',a,')^-1');
{epsilon stays as is}
end;
```

```
procedure dohd(var f : line;
         var last,i,enext : integer);
var
   a,b,u :    string;
   s1,s2,zprev : string[5];
   sa,sb,su,j : integer;

begin
  a:=f[i+1].a[1]; sa:=f[i+1].s[1];
  b:=f[i+1].a[2]; sb:=f[i+1].s[2];
  u:=f[i-1].a[1]; su:=f[i-1].s[1];
  zprev := str(f[i+1].e);
{copy i+2 to last into i+3 to last+1}
  for j := last downto i+2 do
     f[j+1] := f[j];
  last := last + 1;
{move W(u)W(-1) into new position}
 f[i+2] := f[i];
 f[i+1] := f[i-1];
{create a D-S symbol at i-1}
  f[i-1].l := 'd';
  f[i-1].s[1] := su * sa;
  f[i-1].s[2] := su * sb;
  f[i-1].a[1] := concat('(',u,')(',a,')');
  f[i-1].a[2] := concat('(',b,')(',u,')⁻¹');
{create epsilon for <a,b> at i-1}
  f[i-1].e := enext;
  s1 := dash(f[i-1].s[1]);
  s2 := dash(f[i-1].s[2]);
  elist[enext] := concat('1+(', s1, f[i-1].a[1], ')(', s2,
                                                  f[i-1].a[2], ')');
{next enext}
  enext := enext + 1;
{create a Steinberg symbol at i}
```

```
  f[i].l := 's';
  f[i].s[1] := su;
  f[i].s[2] := 1;
  f[i].a[1] := concat('f',zprev,'(',u,')ε̂⁻¹',zprev);
  f[i].a[2] := concat('ε̂⁻¹',zprev);
{create epsilon for symbol at i}
  f[i].e := enext;
  s1 := dash(f[i].s[1]);
  s2 := dash(f[i].s[2]);
  elist[enext] := concat('[', s1, f[i].a[1], ',', s2, f[i].a[2],
                                                                    ']');
{next enext}
  enext := enext + 1;
end;

procedure movexl (var f : line;
         var last, enext : integer;
         var done : boolean);
  var
   i : integer;
   found : boolean;
 begin
  find2('l', f, last, i, found);
  if found then
   case f[i + 1].l[1] of
    '1' :
     writeln(outfile,'error');
    '2' :
     begin
      doxlx2(f, last, i, enext);
      done := false
     end;
    'w' :
     begin
      doxlw(f,last,i,enext);
```

```
      done := false
     end;
    'd' :
     begin
      dox1d(f,last,i,enext);
      done := false
     end;
    's' :
     begin
      dox1s(f,last,i,enext);
      done := false
     end;
  end;
end;

procedure movex2 (var f : line;
        var last, enext : integer;
        var done : boolean);
var
   didone : boolean;
   i : integer;
begin
   didone := false;
   i := last - 1;
   while (i>=1) and not didone do
     begin
       if f[i].1 = '2' then
         case f[i+1].1[1] of
         '1' : ,
         '2' : writeln(outfile,'error');
         'w' : begin
                 dox2w(f,last,i,enext);
                 done := false;
                 didone := true;
               end;
```

```
        'd' : begin
                dox2d(f,last,i,enext);
                done := false;
                didone := true;
              end;
        's' : begin
                dox2s(f,last,i,enext);
                done := false;
                didone := true;
              end;
        end;
        i := i - 1;
    end;
end;

procedure refineww (var f : line;
        var last, enext : integer;
        var done : boolean);
var
   didone: boolean;
   i: integer;
begin
   didone := false;
   i := 1;
   while (i<last) and not didone do
     begin {look for WW that's not a WW(-1)}
       if (f[i].l = 'w') and (f[i+1].l = 'w') and
           ((f[i+1].s[1] <> -1) or
               ((f[i+1].a[1] <>'1') and (f[i+1].a[1] <> '')))
then

          begin
             doww(f,last,i,enext);
             done := false;
             didone := true;
```

```
            end;
        i := i + 1;
      end;
end;

procedure multw (var f : line;
        var last, enext : integer;
        var done : boolean);
var
   didone : boolean;
   i : integer;
begin
   didone := false;
   i := 1;
   while (i<=last-2) and not didone do
     begin
       if (f[i].l[1] = 'w') and (f[i+1].l[1] = 'w') and
(f[i+2].l[1] = 'w') then
         if (f[i+1].s[1]=-1) and ((f[i+1].a[1] = '') or
(f[i+1].a[1] = '1'))
           then begin
                 dowww(f,last,i,enext);
                 done := false;
                 didone := true
               end
           else begin
                 doww(f,last,i,enext);
                 done := false;
                 didone := true
               end;
       i := i + 1;
     end
end;

procedure movew (var f : line;
```

```
        var last, enext : integer;
        var done : boolean);
var
   didone: boolean;
   i: integer;
begin
   didone := false,
   i := 1;
   while (i<last) and not didone do
     begin
      {check 1st for ww(-1)d/ww(-1)s for an hd/hs}
       if (f[i].l[1] = 'w') and (i>1) then
                  if ((f[i].a[1]='') or (f[i].a[1]='1'))
                        and (f[i].s[1]=-1) and (f[i-1].l[1]='w')
                  then
                      if f[I+1].l[1]='d' then begin
                                                dohd(f,last,i,enext),
                                                done := false,
                                                didone := true
                                                 end
                      else if f[i+1].l[1]='s' then begin
                                                 dohs(f,last,i,enext);
                                                 done := false;
                                                 didone := true
                                                      end;
      {check 2nd for a regular wd or ws}
       if (didone = false) and (f[i].l[1] = 'w') then
            begin
               case f[i+1].l[1] of
                'd': begin
                       dowd(f,last,i,enext);
                       done := false,
                       didone := true
                     end;
                's'  begin
```

```
                    dows(f,last,i,enext);
                    done := false;
                    didone := true
                   end;
                end;
              end;
          i := i + 1,
       end;
end;

procedure doxx (var f : line,
        var last, i : integer),    {combine xixi ---> xi}
begin
 f[i].a[1] := concat(f[i].a[1], dashplus(f[i+1].s[1]),
                f[i+1].a[1]);
 i := i + 1,
 last := last - 1,
 while i <= last do
  begin
   f[i] := f[i + 1];
   i := i + 1;
  end,
end,

procedure smash (var f : line;
        var last : integer;
        var done : boolean);
 var
  smashed : boolean;
  i : integer;
begin
 smashed := false;
 repeat
  if pairx1(f, last, i) and (last > 1) then
   begin
```

```
    doxx(f, last, i),
    done := false
   end
  else if pairx2(f, last, i) and (last > 1) then
   begin
    doxx(f, last, i);
    done := false
   end
  else
   smashed := true;
 until smashed
end,

procedure reduce (var f : line;                                  {main
reduction alg}
        var enext, last : integer;
        var noop : boolean);
begin
 noop := true,
 smash(f, last, noop);    {combine all x1's and combine all x2's}
 if noop then movex1(f, last, enext, noop),   {move an x1 to the
                                                      right}
 if noop then movex2(f, last, enext, noop);   {move an x2 to the
                                                      right}
 if noop then multw(f, last, enext, noop);    {multiply leftmost
                                                      three W's}
 if noop then movew(f, last, enext, noop);    {move a W to right of
                                                           a D}
 if noop then refineww(f, last, enext, noop); {WW becomes WW(-1)}
  (* if didn't do anything, then noop := true *)
end;

 procedure printe (var f : line;       {print epsilon number num}
        var num : integer);
begin
```

```
    if copy(elist[num],1,1) = '[' then
       begin
         writeln(outfile,'θ',str(num),' = ',elist[num])
       end
    else
       begin
         writeln(outfile,'ε',str(num),' = ',elist[num])
       end
 end;

procedure printes (var f : line;          {print list of epsilons}
        var efirst,laste : integer);
 var
     j : integer;
 begin
     for j:=efirst to laste do
         printe (f,j);
 end;

 procedure printf (var f : line;          {print out current line}
         var last : integer);
  var
   i : integer;
 begin
  for i := 1 to last do
   case f[i].l[1] of
    '1' : begin
     write(outfile,'X1(', dash(f[i].s[1]), f[i].a[1], ')');
          end;
    '2' : begin
     write(outfile,'X2(', dash(f[i].s[1]), f[i].a[1], ')');
          end;
    'd' : begin
     write(outfile,'<', dash(f[i].s[1]), f[i].a[1], ',',
                               dash(f[i].s[2]), f[i].a[2], '>');
```

```
        end;
    's' : begin
    write(outfile,'(', dash(f[i].s[1]), f[i].a[1], ',',
                                     dash(f[i].s[2]), f[i].a[2], ')');
        end;
    'w' : begin
            if f[i].a[1] = '' then s1 := '1' else s1 := f[i].a[1];
            write(outfile,'W(', dash(f[i].s[1]), s1, ')');
        end
  end;
 writeln; writeln(outfile);
end;

procedure inputf (var f:line;var last,enext,efirst: integer);
 var
    s1,s2 : string[5];
    a1,a2 : string;
    lnum,fop,loc:integer;
 begin
    lnum:=0;
    write('begin with epsilon # ');readln(efirst);enext := efirst;
    repeat
        lnum := lnum + 1;
        writeln('input letter #',lnum:3);
        write('letter: '); readln(f[lnum].l);
        if f[lnum].l <> ' ' then
           if f[lnum].l <> 'r' then
               begin
                  write('(first) argument = ');
                  readln(f[lnum].a[1]);
                  f[lnum].s[1] := takesign(f[lnum].a[1]);
                  if (f[lnum].l = 's') or (f[lnum].l = 'd') then
                     begin
                        write('second argument = ');
                        readln(f[lnum].a[2]);
```

```
                    f[lnum].s[2] := takesign(f[lnum].a[2]);
                    {create epsilon or theta}
                    begin
                      s1 := dash(f[lnum].s[1]);
                      s2 := dash(f[lnum].s[2]);
                      a1 := f[lnum].a[1];
                      a2 := f[lnum].a[2];
                      if f[lnum].l = 's' then
                        elist[enext] :=
                          concat('[',s1,a1,',',s2,a2,']')
                      else
                        elist[enext] :=
                          concat('1+(',s1,a1,')(',s2,a2,')');
                    end;
                    f[lnum].e := enext;
                    enext := enext +1;
                  end;
               if (f[lnum].l = 'w') and (f[lnum].s[1] = -1) and
                                        (f[lnum].a[1] = '1')
                  then f[lnum].a[1] := '';
          end
        else lnum := 0;
    until f[lnum].l = ' ';
    last := lnum-1;
    {get and do the first operation}
    writeln('first op => location (NO COMMAS) : ');
readln(fop,loc);
    case fop of
      4: if ((f[loc].l = '1') and (f[loc+1].l = '1'))
           or ((f[loc].l = '2') and (f[loc+1].l = '2')) then
             begin
               printf(f,last);
               doxx(f,last,loc);
             end
         else writeln('error in input');
```

```
5: if ((f[loc].l = '1') and (f[loc+1].l = 'w')) then
      begin
         printf(f,last);
         dox1w(f,last,loc,enext)
      end
   else if (f[loc].l = '2') and (f[loc+1].l = 'w') then
            begin
               printf(f,last);
               dox2w(f,last,loc,enext)
            end
        else writeln('error in input');
6: if ((f[loc].l = '1') and (f[loc+1].l = '2')) then
      begin
         printf(f,last);
         dox1x2(f,last,loc,enext)
      end
   else writeln('error in input');
8: if (f[loc].l = 'w') and (f[loc+1].l = 'w')
      and ((f[loc+1].s[1] <> -1) or
         ((f[loc+1].a[1] <>'1') and (f[loc+1].a[1] <> ' ')))
   then
      begin
         printf(f,last);
         doww(f,last,loc,enext)
      end
   else writeln('error in input');
9: if (f[loc].l = 'w') and (f[loc+1].l = 'w')
      and (f[loc+1].s[1] = -1)
      and ((f[loc+1].a[1] = ' ') or (f[loc+1].a[1] = '1'))
      and (f[loc+2].l = 'w')
   then
      begin
         printf(f,last);
         dowww(f,last,loc,enext)
      end
```

```
        else writeln('error in input');
     10: if (f[loc].l = '1') and (f[loc+1].l = 'd') then
           begin
              printf(f,last);
              dox1d(f,last,loc,enext)
           end
         else if (f[loc].l = '1') and (f[loc+1].l = 's') then
                 begin
                    printf(f,last);
                    dox1s(f,last,loc,enext)
                 end
         else if (f[loc].l = '2') and (f[loc+1].l = 'd') then
                 begin
                    printf(f,last);
                    dox2d(f,last,loc,enext)
                 end
         else if (f[loc].l = '2') and (f[loc+1].l = 's') then
                 begin
                    printf(f,last);
                    dox2s(f,last,loc,enext)
                 end
         else writeln('error in input');
     11  if (f[loc].l = 'w') and (f[loc+1].l = 'd') then
                 begin
                    printf(f,last);
                    dowd(f,last,loc,enext)
                 end
         else if (f[loc].l = 'w') and (f[loc+1].l = 's') then
                 begin
                    printf(f,last);
                    dows(f,last,loc,enext)
                 end
         else writeln('error in input');
     otherwise writeln('No operation performed before
reduction ');
```

```
       end;
 end;

begin
 writeln('What is the output file''s name: ');
 readln(filename);
 rewrite(outfile,filename);
 enext := 1;
 inputf(f, last, enext, efirst);   {get the target word}
 done := FALSE;
 repeat
  printf(f, last);
  reduce(f, enext, last, done);
 until done;
 laste := enext-1;
 printes(f,efirst, laste);
 close(outfile)
end.
```

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**Title of Dissertation:** ON THE RELATIVE $K_2$ OF NON-COMMUTATIVE LOCAL RINGS

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